DIMENSION IS POLYNOMIAL IN HEIGHT FOR POSETS WITH PLANAR COVER GRAPHS

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Abstract. We show that height $h$ posets that have planar cover graphs have dimension $O(h^6)$. Previously, the best upper bound was $2^{O(h^3)}$. Planarity plays a key role in our arguments, since there are posets such that (1) dimension is exponential in height and (2) the cover graph excludes $K_5$ as a minor.

1. Introduction

In this paper, we study finite partially ordered sets, posets for short, and we assume that readers are familiar with the basics of the subject, including chains and antichains; minimal and maximal elements; height and width; order diagrams (also called Hasse diagrams); and linear extensions. For readers who are new to combinatorics on posets, several of the recent research papers cited in our bibliography include extensive background information.

Following the traditions of the subject, elements of a poset are also called points. Recall that when $P$ is a poset, an element $x$ is covered by an element $y$ in $P$ when $x < y$ in $P$ and there is no element $z$ of $P$ with $x < z < y$ in $P$. We associate with $P$ an ordinary graph $G$, called the cover graph of $P$, defined as follows. The vertex set of $G$ is the ground set of $P$, and distinct elements/vertices $x$ and $y$ are adjacent in $G$ when either $x$ is covered by $y$ in $P$ or $y$ is covered by $x$ in $P$.

Dushnik and Miller [1] defined the dimension of a poset $P$, denoted dim($P$), as the least positive integer $d$ such that there are $d$ linear orders $L_1, \ldots, L_d$ on the ground set of $P$ such that $x \leq y$ in $P$ if and only if $x \leq y$ in $L_i$ for each $i \in \{1, \ldots, d\}$. In general, there are many posets that have the same cover graph, and among them, there may be...
posets which have markedly different values of height, width and dimension. Indeed, it is somewhat surprising that we are able to bound any combinatorial property of a finite poset in terms of graph theoretic properties of its cover graph.

However, Streib and Trotter [11] proved that dimension is bounded in terms of height for posets that have a planar cover graph. This stands in sharp contrast with a number of well-known families of posets that have height 2 but unbounded dimension (e.g. the standard examples discussed below). The result from [11] prompted researchers to investigate in greater depth connections between dimension and graph theoretic properties of cover graphs. Subsequently, it has been shown that dimension is bounded in terms of height for posets whose cover graphs:

- Have bounded treewidth, bounded genus, or more generally exclude an apex-graph as minor [4];
- Exclude a fixed graph as a (topological) minor [17, 10];
- Belong to a fixed class with bounded expansion [7].

Moreover, the existence of bounds for dimension of posets with cover graphs in a fixed class can say something about the sparsity of the class. Joret, Micek, Ossona de Mendez, and Wiechert [3] proved that a monotone class of graphs is nowhere dense if and only if for every $h \geq 1$ and every $\varepsilon > 0$, posets of height $h$ with $n$ elements whose cover graphs are in the class have dimension $O(n^\varepsilon)$.

The best upper bound to date on dimension in terms of height for posets that have planar cover graphs is $2^{O(h^3)}$. This result can be extracted from [3] via connections between dimension for posets and weak-coloring numbers of their cover graphs. We will give additional details on this work in the next section.

Our main theorem improves this exponential bound to one which is polynomial in $h$.

**Theorem 1.** If $P$ is a poset of height $h$ and the cover graph of $P$ is planar, then $\dim(P) = O(h^6)$.

Planarity plays a crucial role in the existence of a polynomial bound. In [6], Joret, Micek and Wiechert show that for each even integer $h \geq 2$, there is a height $h$ poset $P$ with dimension at least $2^{h/2}$ such that the cover graph of $P$ excludes $K_5$ as a minor.

To discuss lower bounds, we pause to give the following construction which first appears in [1]. For each $n \geq 2$, let $S_n$ be the height 2 poset with \{a_1, a_2, \ldots, a_n\} the set of minimal elements, \{b_1, b_2, \ldots, b_n\} the set of maximal elements, and $a_i < b_j$ in $S_n$ if and only if $i \neq j$. Posets in the family $\{S_n : n \geq 2\}$ are now called standard examples, as $\dim(S_n) = n$ for every $n \geq 2$.

To date, the best lower bound for the maximum dimension of a height $h$ poset with a planar cover graph is $2h - 2$, and this bound comes from the “double wheel” construction given in [6], and illustrated here in Figure 1. To avoid clutter, we do not show arrowheads in our figures. Instead, we indicate directions using color and accompanying narrative.
Figure 1. We illustrate the double wheel construction when \( h = 5 \). Note that the elements \( a_1, \ldots, a_{10} \) and \( b_1, \ldots, b_{10} \) induce a standard example, so the dimension of the depicted poset is at least 10. On the other hand, the height of \( P \) is 5.

In this figure, the black edges are oriented in each individual wheel from outside to inside. The elements of \( \{a_1, \ldots, a_n\} \) are minimal elements so the red edges are oriented “left-to-right” and the blue edges are oriented “right-to-left.”

Requiring that the diagram of a poset \( P \) is planar is a stronger restriction than requiring that the cover graph of \( P \) is planar. Recall that in the diagram: elements are drawn as distinct points in the plane, and each cover relation \( a \leq b \) in \( P \) is represented by a curve from \( a \) to \( b \) going upwards. Accordingly among posets that have planar cover graphs, some but not all also have planar order diagrams. Among the class of posets with planar diagrams, Joret, Micek and Wiechert [6] showed that \( \text{dim}(P) \leq 192h + 96 \) when \( P \) has height \( h \).

The remainder of this paper is organized as follows. In the next section, we prove three reductions to simpler problems, and we give essential background material. The proof of Theorem 1 is given in the following two sections, and we close with brief comments on challenging open problems that remain.

2. Preliminary Reductions and Background Material

When \( P \) is a poset, we write \( x \parallel_P y \) (also \( x \parallel y \) in \( P \)) when \( x \) and \( y \) are incomparable. In general, we prefer the short form \( x \parallel_P y \), except when subscripts or primes are involved. A similar remark applies to the relations \(<, >, \leq, \geq\). Later in the proof, we will discuss a poset \( P \) and define linear orders \( T \) and \( S \) on subsets of the ground set of \( P \). In that discussion, we will write \( u <_T v \) or \( u <_S v \), as appropriate.

We list below some elementary, and well known, properties of dimension.

(i) Dimension is monotonic, i.e., if \( Q \) is a subposet of \( P \), then \( \text{dim}(Q) \leq \text{dim}(P) \).

(ii) The dual of a poset \( P \) is the poset \( P' \) on the same ground set of \( P \) with \( x < y \) in \( P' \) if and only \( x >_P y \). Then \( \text{dim}(P) = \text{dim}(P') \).
For the balance of this preliminary section, we fix a poset $P$. We let $\text{Min}(P)$ and $\text{Max}(P)$ denote, respectively, the set of minimal elements and the set of maximal elements of $P$. Also, we let $\text{Inc}(P)$ denote the set of all ordered pairs $(x, y)$ with $x \parallel_P y$. We will assume $\text{Inc}(P) \neq \emptyset$; otherwise $P$ is a chain and $\dim(P) = 1$. When $(x, y) \in \text{Inc}(P)$ and $L$ is a linear extension of $P$, we say that $L$ reverses $(x, y)$ when $x > y$ in $L$. A set $I \subseteq \text{Inc}(P)$ is reversible if there is a linear extension $L$ of $P$ which reverses every pair in $I$. Vacuously, the empty set is reversible. We then define $\dim(I)$ as the least $d \geq 1$ such that $I$ can be covered by $d$ reversible sets. It is easily seen that $\dim(P)$ is equal to $\dim(\text{Inc}(P))$.

Given sets $A, B \subseteq P$, we let $\text{Inc}(A, B)$ be the set of pairs $(a, b) \in \text{Inc}(P)$ with $a \in A$ and $b \in B$. We use the abbreviation $\dim(A, B)$ for $\dim(\text{Inc}(A, B))$. Again, $\dim(A, B) = 1$ when $\text{Inc}(A, B) = \emptyset$. Typically, we will have $A \subseteq \text{Min}(P)$ and $B \subseteq \text{Max}(P)$.

A sequence $((x_1, y_1), \ldots, (x_k, y_k))$ of pairs from $\text{Inc}(P)$ with $k \geq 2$ is an alternating cycle of size $k$ if $x_i \leq_P y_{i+1}$ for all $i \in \{1, \ldots, k\}$, cyclically (so $x_k \leq_P y_1$ is required). Observe that if $((x_1, y_1), \ldots, (x_k, y_k))$ is an alternating cycle in $P$, then any subset $I \subseteq \text{Inc}(P)$ containing all the pairs on this cycle is not reversible; otherwise we would have $y_i <_L x_i \leq_L y_{i+1}$ for each $i \in \{1, \ldots, k\}$ cyclically, which cannot hold.

A sequence $((x_1, y_1), \ldots, (x_k, y_k))$ of pairs from $\text{Inc}(P)$ is a strict alternating cycle if for each $i, j \in \{1, \ldots, k\}$, we have $x_i \leq_P y_j$ if and only if $j = i + 1$ (cyclically). Note that in this case, $\{x_1, \ldots, x_k\}$ and $\{y_1, \ldots, y_k\}$ are $k$-element antichains. Note that in alternating cycles, we allow that $x_i = y_{i+1}$ for some or even all values of $i$.

When a set $S$ is not reversible and contains an alternating cycle, then an alternating cycle of minimum size in $S$ is easily seen to be a strict alternating cycle. The converse is also true, a detail originally observed by Trotter and Moore [14]: A subset $I \subseteq \text{Inc}(P)$ is reversible if and only if $I$ contains no strict alternating cycle.

When $x <_P y$, a sequence $W = (u_0, u_1, \ldots, u_t)$ is called a witnessing path (from $x$ to $y$) when $u_0 = x$, $u_t = y$ and $u_i$ is covered by $u_{i+1}$ in $P$ for each $i \in \{0, 1, \ldots, t - 1\}$.

The following elementary lemma allows us to concentrate our attention on incomparable pairs from $\text{Inc}(\text{Min}(P), \text{Max}(P))$. See for instance [5, Observation 3] for a proof.

**Lemma 2** (Reduction to min-max). For every poset $P$, there is a poset $Q$ containing $P$ as an induced subposet such that

(i) The height of $P$ is the same as the height of $Q$;

(ii) The cover graph of $Q$ is obtained from the cover graph of $P$ by adding some degree 1 vertices; and

\[ \dim(P) \leq \dim(\text{Min}(Q), \text{Max}(Q)). \]

2.1. **Constrained Subsets and Weak-Coloring Numbers.** Let $P$ be a poset. We say that a non-empty subset $I \subseteq \text{Inc}(\text{Min}(P), \text{Max}(P))$ is singly constrained in $P$ when
there is an element $x_0 \in P$ such that $x_0 <_P b$ for every $(a, b) \in I$. To identify the element $x_0$, we will also say $I$ is singly constrained by $x_0$.

The following lemma was used first in [11] for posets with planar cover graphs and in a more complex form in [5]. The underlying principle is the concept of unfolding, which is an analogue of breadth first search for posets.

**Lemma 3** (Reduction to singly constrained). For every poset $P$, there exists a poset $Q$ such that

(i) The height of $Q$ is at most the height of $P$.

(ii) The cover graph of $Q$ is a minor of the cover graph of $P$.

(iii) There is a minimal element $x_0$ in $Q$ such that $x_0 \leq q$ for every $q \in \text{Max}(Q)$, and

$$\dim(\text{Min}(P), \text{Max}(P)) \leq 2 \dim(\text{Min}(Q), \text{Max}(Q)).$$

In particular, the set $I = \text{Inc}(\text{Min}(Q), \text{Max}(Q))$ in the lemma is singly constrained by $x_0$. We point out that the lemma produces an element $x_0 \in \text{Min}(Q)$, but later in this paper, we will be discussing sets $I \subseteq \text{Inc}(\text{Min}(Q), \text{Max}(Q))$ such that $I$ is singly constrained by an element $x_0$ which is not a minimal element in $Q$.

We say that a non-empty subset $I$ of $\text{Inc}(\text{Min}(P), \text{Max}(P))$ is doubly constrained in $P$ when there is a pair $(x_0, y_0)$ such that

(i) $x_0 <_P y_0$,

(ii) $x_0 <_P b$ for every $(a, b) \in I$, and

(iii) $a <_P y_0$ for every $(a, b) \in I$.

As before, we will also say that $I$ is doubly constrained by $(x_0, y_0)$.

We would very much like to reduce to the case where we are bounding $\dim(I)$ when $I \subseteq \text{Inc}(\text{Min}(P), \text{Max}(P))$ is doubly constrained. Unfortunately, Lemma 3 will not be of assistance. Instead, we will use a different reduction, one that will cost us an $O(h^3)$-factor in the final bound.

The *length* of a path in a graph is the number of its edges. For two vertices $u$ and $v$ in a graph $G$, an $u$–$v$ path is a path in $G$ with ends in $u$ and $v$. Let $G$ be a graph and let $\sigma$ be an ordering of the vertices of $G$. For $r \in \{0, 1, 2, \ldots \} \cup \{\infty\}$ and two vertices $u$ and $v$ of $G$, we say that $u$ is weakly $r$-reachable from $v$ in $\sigma$, if there exists an $u$–$v$ path of length at most $r$ such that for every vertex $w$ on the path, $u \leq_\sigma w$. The set of vertices that are weakly $r$-reachable from a vertex $v$ in $\sigma$ is denoted by $\WReach_r[G, \sigma, v]$. The *weak $r$-coloring number* $\wcol_r(G)$ of $G$ is defined as

$$\wcol_r(G) := \min_{\sigma} \max_{v \in V(G)} |\WReach_r[G, \sigma, v]|,$$

where $\sigma$ ranges over the set of all vertex orderings of $G$. We call $\wcol_r(G)$ the $r$-th weak coloring number of $G$. 
Weak coloring numbers were originally introduced by Kierstead and Yang [9] as a generalization of the degeneracy of a graph (also known as the coloring number). Since then, they have been applied in several novel situations (see Zhu [18] and Van den Heuvel et al. [16], for examples). We also have good bounds on weak coloring numbers. For planar graphs, van den Heuvel et al. [15] have shown that the $r$-th weak coloring number is at most $(\frac{r+2}{2}) \cdot (2r + 1) = \mathcal{O}(r^3)$. See also a recent paper [2] with a lower bound in $\Omega(r^2 \log r)$.

Here is a lemma on weak coloring numbers from [3] that will play an important role in the reduction to the doubly constrained case.

**Lemma 4.** Let $P$ be a height $h$ poset, let $G$ be the cover graph of $P$, and let $c := \text{wcol}_{4h-4}(G)$. Then there is an element $z_0 \in P$ such that the set $J = \{(a, b) \in I : a <_P z_0\}$ satisfies
\[
\dim(J) \geq \frac{\dim(I)}{c} - 2.
\]

We then have the following immediate corollary.

**Corollary 5.** Let $P$ be a poset with a planar cover graph, and let $x_0$ be an element of $P$ such that $x_0 < b$ in $P$ for every $b \in \text{Max}(P)$. Let $I$ be a non-empty subset of $\text{Inc}(\text{Min}(P), \text{Max}(P))$. Then there is a set $J \subseteq I$ such that $J$ is doubly constrained in $P$ and
\[
\dim(I) = \mathcal{O}(h^3) \cdot \dim(J).
\]

**Proof.** Let $G$ be the cover graph of $P$. Apply Lemma 4 with $c = \text{wcol}_{4h-4}(G) = \mathcal{O}(h^3)$ to obtain the element $z_0$ and the set $J \subseteq I$. Let $y_0$ be any maximal element with $z_0 \leq y_0$ in $P$. Since $y_0 \in \text{Max}(P)$ we have $x_0 < y_0$ in $P$. Evidently $J$ is doubly constrained by the pair $(x_0, y_0)$. The inequality from Lemma 4 becomes $\dim(I) \leq c \cdot (2 + \dim(J))$, and with this observation, the proof of the corollary is complete. \qed

2.2. A Reduction to Doubly Exposed Posets. Let $P$ be a poset. We will say that a non-empty set $I \subseteq \text{Inc}(\text{Min}(P), \text{Max}(P))$ is doubly exposed in $P$ if the following conditions are met:

(i) $I$ is doubly constrained by $(x_0, y_0)$.

(ii) The cover graph $G$ of $P$ is planar, and there is a plane drawing of $G$ with $x_0$ and $y_0$ on the same face.

Note that in the preceding definition, we could just as well have required that $x_0$ and $y_0$ be on the exterior face. The form of the definition allows us to determine that a set $I$ is doubly exposed as evidenced by a plane drawing with $x_0$ and $y_0$ on the same face. If desired, we can then redraw the cover graph, without edge crossings, so that $x_0$ and $y_0$ are on the exterior face.

Our next goal is to prove a reduction to the doubly exposed case. The argument requires a technical detail regarding paths. When $R$ is a tree in $G$ and $u, v \in R$, we denote by
$uRv$ the unique path in $R$ from $u$ to $v$. This notation is particularly convenient for discussing concatenation of paths, and it will be used extensively later in the paper.

Let $P$ be a poset with a planar cover graph, and suppose that $I$ is singly constrained by $x_0$. Then consider a plane drawing of the cover graph $G$ of $P$ with $x_0$ on the exterior face. Add to the drawing an extra edge linking $x_0$ from an “imaginary point” located in the outer face. Let $M$ be a non-trivial path in $G$ starting from $x_0$, and let $v$ be the other endpoint of $M$. Now let $M'$ be another path in $G$ also starting from $x_0$, sharing some initial segment with $M$, say the portion from $x_0Mu$ with $u \neq v$. Note that $u$ could coincide with $x_0$. Suppose further that the portion of $M'$ after $u$ is non-empty. Since $x_0$ is on the exterior face in the drawing of $G$ (and since we added the imaginary line), there is a natural notion of “sides”, and we can say with precision that either $M'$ leaves $M$ from the left side, or $M'$ leaves $M$ from the right side. Note however that sides are not well defined when $u$ is the last point of $M$. We illustrate these concepts in Figure 2.

With this technical detail in hand, we are ready for the reduction to the doubly exposed case.

**Lemma 6.** Let $P$ be a height $h$ poset with a planar cover graph. If $I \subseteq \text{Inc}(<\text{Min}(P), \text{Max}(P)>)$ is doubly constrained in $P$, then there is a poset $Q$ and a set $J \subseteq \text{Inc}(<\text{Min}(Q), \text{Max}(Q)>)$ such that

(i) the height of $Q$ is at most $h$;
(ii) the cover graph of $Q$ is a subgraph of the cover graph of $P$;
(iii) $J$ is doubly exposed in $Q$; and

$$\dim(I) \leq 2(h - 1) \dim(J).$$

**Proof.** Let $I \subseteq \text{Inc}(<\text{Min}(P), \text{Max}(P)>)$ be a non-empty doubly constrained set by $(x_0, z_0)$, and let $D$ be a plane drawing of the cover graph $G$ of $P$ with $x_0$ on the exterior face.

We fix a chain from $x_0$ to $z_0$ and refer to this chain as the spine. Label the points on the spine as $\{u_0, u_1, \ldots, u_t\}$ with $x_0 = u_0$, $z_0 = u_t$ and $u_i$ covered by $u_{i+1}$ in $P$ for each $i \in \{1, \ldots, t - 1\}$. Note that $t \leq h - 1$. 
Let
\[ A = \{ a \in P \mid \text{there exists } b \text{ such that } (a, b) \in I \}, \]
\[ B = \{ b \in P \mid \text{there exists } a \text{ such that } (a, b) \in I \}. \]

In particular, we have \( I \subseteq A \times B \), so \( \dim(I) \leq \dim(A, B) \).

For each \( b \in B \), let \( \tau(b) \) be the largest integer \( i \) so that \( u_i < P b \). Note that \( 0 \leq \tau(b) \leq t - 1 \).

Let \( W(b) \) be a witnessing path from \( x_0 \) to \( b \) such that \( W(b) \) shares the initial segment \( (u_0, u_1, \ldots, u_{\tau(b)}) \) with the spine.

We partition \( B \) into \( B_{\text{left}} \) and \( B_{\text{right}} \) in such a way that \( b \) is assigned to the set \( B_{\text{left}} \) if \( W(b) \) leaves the spine from the left side. Dually, we assign \( b \) to \( B_{\text{right}} \) if \( W(b) \) leaves the spine from the right side.

For each \( a \in A \), let \( \tau(a) \) be the least integer \( i \) so that \( a < P u_i \). Now we have \( 2 \leq \tau(a) \leq t \).

We partition the set \( A \) into \( A_2 \cup A_3 \cup \cdots \cup A_t \) by assigning \( a \) to \( A_i \) when \( \tau(a) = i \). Clearly,
\[ \dim(I) \leq \dim(A, B) \leq \sum_{s \in \{2, \ldots, t\}} \sum_{\text{dir} \in \{\text{left}, \text{right}\}} \dim(A_s, B_{\text{dir}}). \]

It follows that there is some \( s \in \{2, \ldots, t\} \) and \( \text{dir} \in \{\text{left}, \text{right}\} \) so that
\[ \dim(A_s, B_{\text{dir}}) \geq \frac{\dim(I)}{2(h - 1)}. \]

We assume that \( \text{dir} = \text{right} \). From the details of the argument, it will be clear that the proof is symmetric in the other case.

We say that an edge \( e = u_iv \) in the cover graph of \( P \) is bad if \( 0 \leq i < s \), \( v \) is not on the spine, and the path \( \{u_0, u_1, \ldots, u_i, v\} \) leaves the spine from the left side. We then define a poset \( Q \) having the same ground set as \( P \) with \( x \leq y \) in \( Q \) if and only if there is a witnessing path in \( P \) from \( x \) to \( y \) avoiding bad edges.

We claim that for every \( a \in A_s \) and every \( b \in B_{\text{right}} \), we have \( a \leq b \) in \( Q \) if and only if \( a \leq b \) in \( P \). The forward implication is obvious. To see the backward one, let \( a \in A_s \) and \( b \in B_{\text{right}} \) with \( a < P b \). Then let \( W \) be a witnessing path from \( a \) to \( b \) in \( P \). This path cannot use a bad edge as this would make \( a < u_i \) in \( P \) for some \( i \in \{1, \ldots, s - 1\} \) contradicting \( a \in A_s \). Therefore, the claim holds and also \( \dim(A_s, B_{\text{right}}) \) in \( Q \) is the same as \( \dim(A_s, B_{\text{right}}) \) in \( P \).

Note that the diagram and the cover graph of \( Q \) are obtained simply by removing the bad edges from the diagram and cover graph, respectively, of \( P \). It follows that the cover graph of \( Q \) is planar. Furthermore, \( x_0 \) and \( u_s \) are on the same face, and the set \( \text{Inc}(A_s, B_{\text{right}}) \) is doubly exposed by the pair \( (x_0, u_s) \). With this observation, the proof of the lemma is complete.

\[ \square \]

Summarizing, we can combine Lemma 3, Corollary 5, and Lemma 6 to obtain:
Corollary 7. Let $P$ be a height $h$ poset with a planar cover graph. Then there is a poset $Q$ such that

(i) $Q$ has height at most $h$;
(ii) $Q$ has a planar cover graph;
(iii) There is a set $I \subseteq \text{Inc} (\text{Min}(Q), \text{Max}(Q))$ such that $I$ is doubly exposed in $Q$ and $\dim(P) = \mathcal{O}(h^4) \cdot \dim(I)$.

We are now ready to begin the proof of our main theorem.

3. LARGE STANDARD EXAMPLES IN DOUBLY EXPOSED POSETS

We pause here to make the following important comment: The concept of height plays no role in the arguments given in this section.

Throughout this section, $P$ will denote a poset with a planar cover graph. Also, $I$ will denote a subset of $\text{Inc} (\text{Min}(P), \text{Max}(P))$ which is doubly exposed by $(x_0, y_0)$. Let

$$A_I = \{ a \in P \mid \text{there exists } b \text{ such that } (a, b) \in I \},$$
$$B_I = \{ b \in P \mid \text{there exists } a \text{ such that } (a, b) \in I \}.$$ 

In particular, we have $I \subseteq A_I \times B_I$.

We will then fix a plane drawing $D$ of $G$, the cover graph of $P$, with $x_0$ and $y_0$ on the exterior face. Next, we discuss a subgraph $T$ of $G$ associated with $x_0$ and the elements of $B_I$. Subsequently, this discussion will be repeated for $y_0$ and the elements of $A_I$.

It is easy to see that there is a subgraph $T$ of $G$ satisfying the following properties.

(i) The vertices and edges of $T$ form a tree containing $x_0$ and all elements of $B_I$.
(ii) The leaves of $T$ are the elements of $B_I$.
(iii) We consider $x_0$ as the root of $T$, and for each $b \in B_I$, we let $x_0 T b$ denote the unique path in $T$ from $x_0$ to $b$. We require that $x_0 T b$ be a witnessing path from $x_0$ to $b$.

We choose and fix a tree $T$ satisfying these properties. In the remainder of the discussion, we will refer to $T$ as the blue tree. The vertices and edges of $T$ are called blue vertices and blue edges respectively. The fact that $x_0$ is on the exterior face implies that $T$ determines a clockwise linear order $\prec_T$ on the elements of $B_I$. We illustrate the notion of a blue tree in Figure 3 where we take $B_I = \{1, \ldots, 15\}$. The leaves have been labeled so that the clockwise order agrees with the natural order as integers. Note that in general, there are many elements of $T$ that do not belong to $\{x_0\} \cup B_I$. Also, there are many elements of $P$ that do not belong to $T$.

In an entirely analogous manner, we determine a red tree $S$ with $y_0$ as its root and the elements of $A_I$ as its leaves. For each $a \in A_I$, we let $a S y_0$ denote the unique path in $S$.
Figure 3. The leaves of $T$ are $\{1, \ldots, 15\}$, the elements of $B_I$. They are ordered clockwise by $<_T$. Recall that we are working in a planar cover graph setup (and not necessarily planar diagram) so the poset relation does not have to go vertically upwards in the plane.

from $a$ to $y_0$, and we require that $aSy_0$ be a witnessing path. Once the red tree $S$ has been chosen, we have a clockwise order $<_S$ on the elements of $S$.

When $C$ is a simple closed curve in the plane, it splits the points of the plane not on $C$ into those that are in the interior of the region bounded by $C$ and those in the exterior of this region. In the discussion to follow, we will abuse terminology slightly and say that a point not on $C$ is either in the interior of $C$ or it is in the exterior of $C$, dropping the reference to the region bounded by $C$.

We find it convenient to assume that in the plane drawing $\mathbb{D}$ of $G$, the vertices $x_0$ and $y_0$ are the lowest, respectively the highest, elements of $P$ in the plane. The entire diagram will be enclosed in a simple closed curve $C$ which intersects $\mathbb{D}$ only at $x_0$ and $y_0$. Note that $x_0$ and $y_0$ are on $C$. All other vertices and edges of the cover graph $G$ are in the interior of $C$. If we start at $x_0$ and traverse the boundary of $C$ in a clockwise direction, we refer to the portion of $C$ between $x_0$ and $y_0$, as the left side of $C$. Continuing on from $y_0$ to $x_0$, we are then on the right side of $C$. If $N$ is any path in $G$ from $x_0$ to $y_0$, then $N$ splits the region bounded by $C$ into two subregions, called naturally, the left half and the right half.

With the set $I$ fixed, with each pair $(a,b) \in A_I \times B_I$ such that $a <_P b$, we will associate a separating path $N = N(a,b)$ from $x_0$ to $y_0$ defined as follows: (1) $u = u(a,b)$ is the least element in $P$ that is on the blue path $x_0Tb$ and satisfies $a <_P u$; (2) $v = v(a,b)$ is the largest element of $P$ that is on the red path $aSy_0$ and satisfies $u \leq_P v$; (3) $N = x_0TuWvSy_0$, where $W$ is an arbitrary witnessing path from $v$ to $u$. Strictly speaking, the path $N(a,b)$ depends on $a$ and $b$ as well as the choice for $W$. However, that detail can be safely ignored, as none of the results to follow depend on which choice is made for $W$.

The path $x_0Tu$ will be called the blue part of $N$; the path $uWv$ will be called the black part of $N$; and the path $vSy_0$ will be called the red part of $N$. We note that the red and black parts share a point, as do the blue and black parts. The black part may consist of a single point, but the red part and the blue part are always non-trivial. We also note that a point on the red part of $N$ may be an element of the blue tree. Analogous comments hold for the other parts of $N$. In general, the vertices $a$ and $b$ do not have to
In the figure on the left, we illustrate a separating path $N = N(a, b)$, with its blue, red, and black parts colored appropriately. In the figure on the right, we show a separating path $N = N(a_4, b)$ and four points in $S$ with $a_1 \leq_s a_2 \leq_s a_3 \leq_s a_4$. Since $a_1$ and $a_3$ are left of $N$, Proposition 8 implies that each of the paths $y_0 S a_1$ and $y_0 S a_3$ contains a point from the union of the black part of $N$ and the blue part of $N$.

When $N = N(a, b)$ is a separating path, we consider the two halves of the region $C$ determined by $N$. When $u$ is a point of $P$, and $u$ is not on $N$, then we will simply say $u$ is left of $N$ when $u$ is in the left half of $F$; analogously, we will say that $u$ is right of $N(a, b)$ when $u$ is in the right half of $C$. Our convention regarding the labeling of the two halves is illustrated in first figure in 4.

The following elementary proposition has four symmetric statements: two for the tree $S$ and two for the tree $T$.

**Proposition 8.** Let $N = N(a, b)$ be a separating path. If $a' \in A_I$, $a' \leq_s a$ and $a'$ is left of $N$, then $y_0 S a'$ contains a point of $N$ from the union of the blue part and the black part of $N$.

**Proof.** If $v(a, b) \in y_0 S a'$, then the proposition holds, since $v(a, b)$ belongs to the black part of $N$. So we may assume $v(a, b) \notin y_0 S z'$. Let $v'$ be the least point of $P$ common to $y_0 S a$ and $y_0 S a'$. Then $v(a, b) <_P v'$. Let $w$ be the first vertex on $y_0 S a'$ after $v'$. Since $a' \leq_s a$, we know $w$ is right of $N$. Since $a'$ is left of $N$, the path $w S a'$ must intersect $N$. Since $S$ is a tree, any point common to $N$ and $w S a'$ belongs to the union of the blue and black parts of $N$. \[\square\]

The next proposition is actually an immediate corollary of Proposition 8. It is stated for emphasis. Note that that there is dual form.
Proposition 9. Let \( N = N(a, b) \) be a separating path. If \( a' \in A_I \) and \( a' \parallel_P b \), then \( a' \) is right of \( N \) if and only if \( a' <_S a \). Also, if \( b' \in B_I \) and \( b' \parallel_P a \), then \( b' \) is left of \( N \) if and only if \( b' <_T b \).

Proposition 10. Let \( N = N(a, b) \) be a separating path. If \( w <_P z \), \( w \) is on one side of \( N \) and \( z \) is on the other, then either \( w <_P b \) or \( a <_P z \).

Proof. Let \( W \) be a witnessing path from \( w \) to \( z \). Then \( W \) and \( N \) must intersect. Let \( q \) be a common point. If \( q \) is on the blue part of \( N \), then \( w <_P b \). If \( q \) is on the red part of \( N \), then \( a <_P z \). If \( q \) is on the black part of \( N \), then both \( w <_P b \) and \( a <_P z \) hold. \( \square \)

Let \( b \) and \( b' \) be distinct elements of \( B_I \). We say that \( b \) is enclosed by \( b' \) if there is a cycle \( D \) in \( G \) such that (1) all points of \( D \) belong to \( D_P[b] \); and (2) \( b \) is in the interior of \( D \). Dually, if \( a \) and \( a' \) are distinct points of \( A_I \), we say that \( a \) is enclosed by \( a' \) if there is a cycle \( D \) in \( G \) such that (1) all points of \( D \) belong to \( U_P[a'] \); and (2) \( a \) is in the interior of \( D \).

Proposition 11. Let \( k \geq 2 \) and let \( ((a_1, b_1), \ldots, (a_k, b_k)) \) be a strict alternating cycle of pairs from \( I \). If \( i, j \) are distinct integers from \([k]\), then \( b_i \) is not enclosed by \( b_j \), and \( a_i \) is not enclosed by \( a_j \).

Proof. We prove that if \( i, j \) are distinct integers from \([k]\), then \( b_i \) is not enclosed by \( b_j \). The proof of the second assertion is symmetric.

Let \( D \) be a cycle in \( G \) evidencing that \( b_i \) is enclosed in \( b_j \). Since \( y_0 \) is on the exterior face, \( y_0 \) is not in the interior of \( D \). Note that \( a_{i-1} <_P b_i \), and \( a_{i-1} \parallel_P b_j \). If \( a_{i-1} \) is in the interior of \( D \), then a witnessing path \( W = W(a_{i-1}, y_0) \) contains a point \( w \) from \( D \). This implies \( a_{i-1} <_P w <_P b_j \). In turn, this implies \( a_{i-1} <_P b_j \), which is false. The contradiction shows that \( a_{i-1} \) is not in the interior of \( D \).

Now consider a witnessing path \( W' = W'(a_{i-1}, b_i) \). Since \( b_i \) is in the interior of \( D \), \( W' \) contains a point \( w' \) from \( D \). This implies \( w' \leq_P b_j \). In turn, we have \( a_{i-1} \leq_P w' \leq_P b_j \). Again, this implies \( a_{i-1} <_P b_j \). The contradiction completes the proof. \( \square \)

Proposition 12. If \( ((a_1, b_1), (a_2, b_2)) \) is an alternating cycle of pairs from \( I \), then \( a_1 <_S a_2 \) if and only if \( b_1 <_T b_2 \).

Proof. We assume that \( a_1 <_S a_2 \) and \( b_2 <_T b_1 \) and show that this leads to a contradiction. Let \( N = N(a_1, b_2) \) be a separating path. Since we are assuming \( b_2 <_T b_1 \) and \( a_1 \parallel_P b_1 \), it follows by Proposition 9 that \( b_1 \) is right of \( N \). Since \( a_2 >_S a_1 \) and \( a_2 \parallel_P b_2 \), again by Proposition 9, we know \( a_2 \) is left of \( N \). Applying Proposition 10 for \( w = a_2 \) and \( z = b_1 \), we conclude that either \( a_1 <_P b_1 \) or \( a_2 <_P b_2 \), but both of these statements are false. The contradiction completes the proof. \( \square \)

We illustrate the implications of Proposition 12 on the left side of Figure 5.
For a non-empty subset $I \subseteq \text{Inc}(\text{Min}(P), \text{Max}(P))$, we define an auxiliary digraph $H_I$ whose vertex set is $\text{Inc}(A_I, B_I)$. In $H_I$, we have a directed edge from $(a, b)$ to $(a', b')$ when these two pairs form an alternating cycle and $a <_S a'$ (therefore, $b <_T b'$ by Proposition 12). The next proposition implies a notion of transitivity for directed paths in $H_I$, and this concept will prove to be fundamentally important.

**Proposition 13.** Let $n \geq 3$ and let $((a_1, b_1), \ldots, (a_n, b_n))$ be a directed path in $H_I$. Then $((a_i, b_i), (a_j, b_j))$ is an edge in $H_I$ for all $i, j$ with $1 \leq i < j \leq n$. In particular, these pairs form a copy of the standard example $S_n$.

**Proof.** Using induction, it is clear that the lemma holds in general if it holds when $n = 3$. Since $a_1 <_S a_3$ (and $b_1 <_T b_3$), it suffices to show that $a_1 <_P b_3$ and $a_3 <_P b_1$.

We suppose first that $a_1 \parallel_P b_3$ and argue to a contradiction. A symmetric argument shows that the assumption that $a_3 \parallel_P b_3$ leads to a contradiction.

Let $N = N((a_2, b_3)$ be a separating path. Since $b_2 \parallel_P a_2$ and $b_2 <_T b_3$, Proposition 9 implies that $b_2$ is left of $N$. Since $a_1 <_S a_2$, and $a_2 \parallel_P b_3$, Proposition 9 also implies $a_1$ is right of $N$. Since $a_1 <_P b_2$, Proposition 10 implies that either $a_2 < b_2$ or $a_1 <_P b_3$. Since the first option is false, and the second is assumed to be false, we have reached a contradiction.

The argument for the preceding proposition is illustrated on the right side of Figure 5.

For a non-empty subset $I \subseteq \text{Inc}(\text{Min}(P), \text{Max}(P))$, we define $\rho(I)$ to be the maximum size (number of vertices) of a directed path in $H_I$. The proof of the following lemma is (essentially) the same as the argument given for Lemma 5.9 in [11], although we are working here in a more general setting.
Lemma 14. Let \( P \) be a poset with a planar cover graph, and let \( I \subseteq \text{Inc}(\text{Min}(P), \text{Max}(P)) \) be doubly exposed. Then
\[
\dim(I) \leq \rho(I)^2.
\]
In particular, if \( k \) is the largest size of a standard example in \( P \), then \( \dim(I) \leq k^2 \).

Proof. We show \( \dim(I) \leq \rho(I)^2 \) by exhibiting a partition of \( I \) into \( \rho(I)^2 \) reversible sets. These sets will have the form \( I(m, n) \) where \( 1 \leq m, n \leq \rho(I) \). A pair \((a, b) \in I\) belongs to \( I(m, n) \) if

(i) the longest directed path in \( H_I \) starting from \((a, b)\) has size \( m \), and
(ii) the longest directed path in \( H_I \) ending at \((a, b)\) has size \( n \).

To complete the proof, it suffices to show that each \( I(m, n) \) is reversible. We argue by contradiction.

Suppose that for some pair \((m, n)\), the set \( I(m, n) \) is not reversible. Therefore there is a strict alternating cycle \(((a_1, b_1), \ldots, (a_k, b_k))\) of size \( k \geq 2 \) with all pairs from \( I(m, n) \). Without loss of generality, \( a_1 \prec_S a_i \) for each \( i \in \{2, \ldots, k\} \).

If \( k = 2 \), then there is a directed edge from \((a_1, b_1)\) to \((a_2, b_2)\) in \( H_I \). It follows that any directed path in \( H_I \) starting at \((a_2, b_2)\) can be extended by prepending \((a_1, b_1)\). Thus \((a_1, b_1), (a_2, b_2)\) cannot both belong to \( I(m, n) \). We conclude that \( k \geq 3 \).

The balance of the proof divides into two cases. In view of our assumptions regarding the labeling of the pairs in the alternating cycle, exactly one of the following two cases is applicable (see Figure 6):

\[
a_1 \prec_S a_k \prec_S a_2 \quad \text{or} \quad a_1 \prec_S a_2 \prec_S a_k.
\]
We have already noted that \((a_1, b_1)\). In the first case, we will show that there is a directed path in \(H_I\) of size \(m + 1\) starting at \((a_1, b_1)\). In the second case, we will show that there is a directed path in \(H_I\) of size \(n + 1\) ending at \((a_2, b_2)\). Both implications are contradictions. We will give details of the proof for the first case. It will be clear that the argument for the second case is symmetric.

Therefore we assume \(a_1 < \mathcal{S} a_k < \mathcal{S} a_2\). Since the pairs \((a_1, b_1), (a_k, b_2) \in \text{Inc}(A_I, B_I)\) form an alternating cycle of size 2 and \(a_1 < \mathcal{S} a_k\), we have an edge in \(H_I\) from \((a_1, b_1)\) to \((a_k, b_2)\). Since \((a_1, b_1)\) is the first vertex on this edge, we know \(m \geq 2\). By Proposition 12, we have \(b_1 < \mathcal{T} b_2\). Similarly, there is a directed edge in \(H_I\) from \((a_k, b_3)\) to \((a_2, b_1)\), and \(b_3 < \mathcal{T} b_1\). Therefore,

\[ b_3 < \mathcal{T} b_1 < \mathcal{T} b_2. \]

Fix a directed path \(((w_1, z_1), (w_2, z_2), \ldots, (w_m, z_m))\) in \(H_I\) with \((w_1, z_1) = (a_2, b_2)\). (Recall that \(m \geq 2\).) Now consider the sequence

\[ ((a_1, b_1), (a_k, b_2), (w_2, z_2), \ldots, (w_m, z_m)). \]

We claim that this sequence is a directed path in \(H_I\). Since it has size \(m + 1\) and it starts at \((a_1, b_1)\), this will be a contradiction.

We have already noted that \(((a_1, b_1), (a_k, b_2))\) is an edge in \(H_I\) and since all \(((w_i, z_i), (w_{i+1}, z_{i+1}))\) are edges in \(H_I\) as well, it remains only to show that there is an edge from \((a_k, b_2)\) to \((w_2, z_2)\) in \(H_I\). Note that \(a_k < \mathcal{S} a_2 < \mathcal{S} w_1 < \mathcal{S} w_2\), and \(w_2 < \mathcal{P} z_1 = b_2\). Therefore, we only need to show that \(a_k < \mathcal{P} z_2\). We assume that \(a_k \parallel \mathcal{P} z_2\) and show that this leads to a contradiction. Let \(N = N(a_k, b_1)\) be a separating path (see Figure 7). Since \(a_2 \parallel \mathcal{P} b_1\), we know from Proposition 9 that \(a_2\) is left of \(N\). Note also that \(b_1 < \mathcal{T} b_2 = z_1 < \mathcal{T} z_2\). With our assumption that \(a_k \parallel \mathcal{P} z_2\), we know from Proposition 9 that \(z_2\) is right of \(N\). Since \(a_2 = w_1 < \mathcal{P} z_2\), it follows from Proposition 10 that either \(a_2 < \mathcal{P} b_1\) or \(a_k < \mathcal{P} z_2\). The first option is false, and the second is assumed false. Again, we have reached a contradiction. \(\square\)
When $I$ is doubly exposed, we now have $\dim(I)$ bounded in terms of $\rho(I)$, independent of the height $h$ of $P$. Now we turn our attention to bounding $\rho(I)$ in terms of $h$.

4. Restrictions Resulting from Bounded Height

This section is devoted to proving the following lemma.

**Lemma 15.** Let $P$ be a height $h$ poset with a planar cover graph. Let $I \subseteq \text{Inc}(\text{Min}(P), \text{Max}(P))$ be doubly exposed in $P$. Then

$$\rho(I) \leq 34h + 11.$$ 

Once this lemma has been proven, the proof of our main theorem will be complete. To see this, recall that using Corollary 7, we paid a price of $O(h^4)$ to reduce to the case where we need to bound $\dim(I)$ for $I$ doubly exposed in $P$. Lemma 14 asserts that $\dim(I) \leq \rho(I)^2$. Combining this with Lemma 15, we obtain the bound $O(h^6)$.

Our final bound on $\rho(I)$ will emerge from a series of preliminary results. In their presentation, we aim for reasonable multiplicative constants and consciously tolerate less than optimal additive constants.

As we did in the last section, we will present a series of small propositions all working within the following context. We fix an integer $h \geq 2$ and assume that we have a poset $P$ whose height is at most $h$. We let $G$ denote the cover graph of $P$. We assume that we have a set $I \subseteq \text{Inc}(\text{Min}(P), \text{Max}(P))$ which is doubly exposed by $(x_0, y_0)$, and we have a plane drawing of $G$ with $x_0$ and $y_0$ on the exterior face. Finally, we have an integer $n \geq 2$ and a directed path $((a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n))$ in $H_I$.

As before, we choose a blue tree $T$ and a red tree $S$. We then have linear orders $<_T$ and $<_S$ on $B_I$ and $A_I$, respectively.

The next proposition is concerned with how a witnessing path $W$ can intersect paths of the form $aSy_0$, where $a$ is an element of the red tree $S$. There is an analogous version for paths of the form $x_0Tb$ where $b$ is a point in the blue tree $T$. When $i \in [n]$, we will say that a witnessing path $W$ cuts $a_i$ when $W$ intersects the path $a_iS_y_0$. Also, when $s_i$ is a point common to $W$ and $a_iS_y_0$, we will say $W$ cuts $a_i$ at $s_i$. When $1 \leq i < j < k \leq n$, $W$ cuts $a_i$ at $s_i$, $W$ cuts $a_k$ at $s_k$, and $s = s_i = s_k$, then the fact that $S$ is a tree implies that $W$ cuts $a_j$ at $s$. When $s_i \neq s_k$, the path $W$ may cut each of $a_iS_y_0$ and $a_kS_y_0$ at several other points. Subsequently, there may be several regions in the plane formed by portions of $a_iS a_k$ and $W$.

The following elementary proposition will play a key role in subsequent arguments. There are actually two versions, one for the red tree $S$ and one for the blue tree $T$. The impact of the proposition is illustrated in Figure 8.

**Proposition 16.** Let $i$ and $k$ be integers with $1 \leq i \leq k \leq n$. If $W$ is a witnessing path, and $W$ cuts $a_i$ and $a_k$, then $W$ cuts $a_j$ for every $j$ with $i < j < k$. Suppose further
that \( W \) cuts \( a_i \) at \( s_i \), \( W \) cuts \( a_k \) at \( s_k \), and \( s_i \leq_P s_k \) (\( s_i \geq_P s_k \), respectively). Then for every \( j \) with \( i < j < k \), there is an element \( s_j \) of \( P \) such that \( W \) cuts \( a_j \) at \( s_j \), and \( s_i \leq_P s_{i+1} \leq_P \cdots \leq_P s_k \) (\( s_i \geq_P s_{i+1} \geq_P \cdots \geq_P s_k \), respectively). Also, if the paths \( s_i W_s k \) and \( s_i S k \) form a simple closed curve \( \mathcal{R} \). Then \( a_j \) is in the exterior of \( \mathcal{R} \) for every \( j \) with \( i < j < k \).

**Proof.** If \( s_i = s_k \), then \( s_i \) belongs to \( a_j S y_0 \) for every \( j \) with \( i \leq j \leq k \). In this case, the final statement of the proposition holds vacuously. Now suppose that \( s_i \neq s_k \) and say \( s_i < s_k \) in \( P \). When \( s_i > s_k \) in \( P \) the argument is symmetric.

It is easy to see that there are points \( s_i' \) and \( s_k' \) on \( W \) such that \( s_i \leq_P s_i' \leq_P s_k' \leq_P s_k \); and the paths \( s_i' W s_k' \) and \( s_i S s_k' \) form a simple closed curve \( \mathcal{R} \) in the plane. First, let \( j \) be any integer with \( i < j < k \). Since \( a_i \leq s_i < a_j < s_k \), either (1) \( a_j \) is in interior of \( \mathcal{R} \); or (2) \( a_j \) is in the exterior of \( \mathcal{R} \) and \( W \) cuts \( a_j \) at a point which belongs to \( s_i' W s_k' \). Now assume that option (1) holds. Since \( a_i \leq_P s_i < a_j < s_k' \), all points on the boundary of \( \mathcal{R} \) belong to \( U_P[a_j] \). This implies that \( a_j \) is enclosed by \( a_i \), which contradicts Proposition 11. We conclude that option (2) must hold. Note that this proof verifies the final statement of the proposition in the special case where \( s_i = s_i' \) and \( s_k' = s_k \).

The remaining part of the proof is a simple inductive argument. There is nothing to prove if \( k = i + 1 \). When \( k = i + 2 \), there is only one valid choice for \( j \), i.e. \( j = i + 1 \), and the previous paragraphs proves the statement. Now suppose that \( k > i + 2 \). The path \( W \) can cut \( a_{i+1} \) in many different places, but we simply fix such an element \( s_i = s_k \) common to \( W \) and \( a_{i+1} S y_0 \). With this choice, we have \( s_i \leq_P s_i = s_k \). If \( s_i = s_k \), we simply take \( s_j = s_k \) for all \( j \) with \( i + 1 < j < k \). If \( s_i = s_k \), then \( s_{i+1} < s_k \), and we apply the proposition to the witnessing path \( s_{i+1} W s_k \) which cuts \( a_{i+1} \) and \( a_k \). \( \square \)
For a non-empty subset $X \subseteq [n]$, we let $A(X) = \{a_i : i \in X\}$ and $B(X) = \{b_i : i \in X\}$. Note that the pairs in $\{(a_i, b_i) : i \in X\}$ determine a directed path of size $|X|$ in $H_f$. Let $\alpha, \beta \in [n] - X$ with $\alpha \neq \beta$, and let $N = N(a_\alpha, b_\beta)$ be a separating path. We will say that $N$ separates $A(X)$ from $B(X)$ if all points of $A(X)$ are on one side of $N$ and all points of $B(X)$ are on the other side.

We present the first of three key results bounding $\rho(I)$ in terms of the height of $P$.

**Proposition 17.** Let $X$ be a non-empty subset of $[n]$, let $\alpha, \beta$ be two distinct integers in $[n] - X$, and let $N = N(a_\alpha, b_\beta)$. Suppose further that $N$ separates $A(X)$ from $B(X)$.

(i) If the black portion of $N$ is trivial, then $|X| = 1$.
(ii) If the black portion of $N$ is non-trivial, then $|X| \leq 2h - 1$.

**Proof.** We give the argument when the points of $A(X)$ are left of $N$ and the points of $B(X)$ are right of $N$. The argument when the sides are reversed is symmetric.

Let $W$ be the black portion of $N$. We assume first that $W$ is trivial. Then $N$ is a witnessing path from $x_0$ to $y_0$, so that the elements on $N$ form a chain in $P$. Now assume that $|X| \geq 2$, and let $a_i$ and $a_j$ be distinct elements of $A(X)$. Then let $W(a_i, b_j)$ and $W(a_j, b_i)$ be arbitrary witnessing paths. Since $a_i$ and $a_j$ are left of $N$, while $b_i$ and $b_j$ are right of $N$, there must be a point $z$ common to $W(a_i, b_j)$ and $N$ and a point $z'$ common to $W(a_j, b_i)$ and $N$. However, $N$ is a chain, so $z$ and $z'$ are comparable in $P$. If $z \leq_P z'$, then $a_i < z \leq z' < b_i$ in $P$, which is false. A similar contradiction is reached if $z' <_P z$. We conclude that $|X| = 1$. This observation completes the proof of the first assertion.

We now assume that $W$, the black portion on $N$, is non-trivial. Consider the red portion of $N$. Clearly, it is a chain on at most $h$ elements from $v(a_\alpha, b_\beta)$ to $y_0$. For each $a \in A(X)$ let $\tau(a)$ be the lowest element in this chain such that $a \leq_P \tau(a)$. Consider also the blue portion of $N$ which is a chain on at most $h$ elements from $x_0$ to $u(a_\alpha, b_\beta)$. For each $b \in B(X)$ let $\tau(b)$ be the highest element of this chain such that $\tau(b) \geq_P b$.

We claim that when $i, j \in X$ and $i < j$, then $\tau(a_i) \leq_P \tau(a_j)$. Assume the contrary, $\tau(a_i) \geq_P \tau(a_j)$. Consider a witnessing path from $a_j$ to $\tau(a_j)$. By our assumption this path must avoid $a_\alpha S y_0$. Thus, we have $\alpha < i < j$ and a witnessing path from $a_j S y_0$ to $a_\alpha S y_0$ avoiding $a_\alpha S y_0$. This is a contradiction with the statement of Proposition 16. Similarly, when $i, j \in X$ and $i < j$, we have $\tau(b_i) \geq_P \tau(b_j)$.

We claim that at least one of these two inequalities $\tau(a_i) \leq_P \tau(a_j)$, $\tau(b_i) \geq_P \tau(b_j)$ must be strict. To see this assume that $\tau(a_i) = \tau(a_j)$ and $\tau(b_i) = \tau(b_j)$. Consider a witnessing path $W'$ from $a_j$ to $b_i$. Since $a_j$ is left of $N$ and $b_i$ is right of $N$, we know that $W'$ intersects $N$. Let $z'$ be a common point of $W'$ and $N$. If $z'$ lies on the red portion of $N$, then $b_i > z' \geq \tau(a_j) = \tau(a_i) \geq a_i$ in $P$ which is a contradiction. If $z'$ lies on the blue portion of $N$, then $a_j < z' \leq \tau(b_i) = \tau(b_j) \leq b_j$ in $P$ which is a contradiction. Thus $z'$ is black. Similarly a witnessing path $W''$ from $a_i$ to $b_j$ must intersect $N$ at a point $z''$ which is black. Since the black portion is a chain, $z'$ and $z''$ are comparable in $P$. If
Consider \( N = N(a_3, b_3) \). The figure suggests that the black part of \( N \) intersects \( a_3S_{y_0} \). This forces \( a_4, a_5, a_6, a_7 \) to be left of \( N \). Similarly, the black part of \( N \) intersects \( x_0T_{b_{11}} \). This forces \( b_{10} \) to be left of \( N \).

If \( z' \preceq_P z'' \), then \( a_j \leq z' \leq z'' \leq b_j \) in \( P \). If \( z' \succeq_P z'' \), then \( a_i \leq z'' \leq z' \leq b_i \) in \( P \). Both statements are false. This observation confirms our claim.

Consider the following two sets \( \{ \tau(a) \mid a \in A(X) \} \), \( \{ \tau(b) \mid b \in B(X) \} \). Each of these can be considered as a sequence sorted by the linear order on \( X \) as a set of integers. The first sequence is non-decreasing on the red chain in \( N \). The second sequence is non-increasing on the blue chain in \( N \). For each consecutive pair \( i < j \) of integers in \( X \) (i.e. there is no \( i' \in X \) with \( i < i' < j \)), we have a change in at least one of the two sequences. Therefore,

\[
2h \geq |\{ \tau(a) \mid a \in A(X) \}| + |\{ \tau(b) \mid b \in B(X) \}| \geq 2 + (|X| - 1).
\]

With this observation, the proof is complete. \( \square \)

**Proposition 18.** If \( T \) and \( S \) have no common vertices, then \( n \leq 6h + 1 \).

**Proof.** We assume that \( n \geq 6h + 2 \) and argue to a contradiction. Let \( N = N(a_{4h}, b_{4h+1}) \) be a separating path. Then set \( u = u(a_{4h}, b_{4h+1}) \), \( v = v(a_{4h}, b_{4h+1}) \). We note that \( v \) belongs to \( a_{4h}S_{y_0} \) but not \( a_{4h+1}S_{y_0} \). Dually, \( u \in x_0T_{b_{4h+1}} \) but not \( x_0T_{b_{4h}} \). Let \( W \) be the black portion of \( N \). Note that \( W \) is non-trivial. We split the elements of the pairs into \( A_1 = \{ a_1, a_2, \ldots, a_{4h-1} \} \), \( A_2 = \{ a_{4h+2}, a_{4h+3}, \ldots, a_{6h+2} \} \), \( B_1 = \{ b_1, b_2, \ldots, b_{4h-1} \} \) and \( B_2 = \{ b_{4h+2}, b_{4h+3}, \ldots, b_{6h+2} \} \).

Proposition 9 implies that both \( a_{4h+1} \) and \( b_{4h} \) are left of \( N \). Propositions 8 and 16 together imply that all elements of \( A_2 \cup B_1 \) are left of \( N \). On the other hand, elements of \( A_1 \cup B_2 \) may be on either side of \( N \).

We illustrate the path \( N \) and possible intersections with paths in \( T \) and \( S \) in Figure 9.
We partition the set \( \{1, 2, \ldots, 4h - 1\} \) as \( X_1 \cup X_2 \), where \( i \in X_1 \) if and only if \( a_i \) is left of \( N \). Since \( N \) separates \( A(X_2) \) from \( B(X_2) \subseteq B_1 \), it follows from Proposition 17 that \( |X_2| \leq 2h - 1 \). Therefore \( |X_1| \geq 4h - 1 - (2h - 1) = 2h \). Similarly, we partition \( \{4h + 2, 4h + 3, \ldots, 6h + 2\} \) as \( Y_1 \cup Y_2 \), where \( i \in Y_1 \) if and only if \( b_i \) is left of \( N \). Now we conclude that \( |Y_2| \leq 2h - 1 \) and therefore \( |Y_1| \geq 2h + 1 - (2h - 1) = 2 \). Set \( m = 2h \). Now we are going to discard excess elements and relabel those that remain.

Let \( A' \) be a subset of \( A(X_1) \) of size \( m \) with elements relabeled as \( \{w_1, \ldots, w_m\} \) so that \( w_1 < S \cdots < S w_m \). Let \( B' \) be the corresponding subset of elements of \( B(X_1) \) with elements relabeled correspondingly as \( \{z_1, \ldots, z_m\} \). Let \( \{z_{m+1}, z_{m+2}\} \) be a subset of \( B(Y_1) \) of size 2 so that \( z_{m+1} < T z_{m+2} \). Let \( \{w_{m+1}, w_{m+2}\} \) be the corresponding subset of elements of \( A(Y_1) \). Note that we have

\[
\begin{align*}
w_1 &< S \cdots < S w_m < S a_{4h} < S a_{4h+1} < S w_{m+1} < S w_{m+2}, \\
z_1 &< T \cdots < T z_m < T b_{4h} < T b_{4h+1} < T z_{m+1} < T z_{m+2}.
\end{align*}
\]

Let \( N' = N(w_{m+1}, z_{m+2}) \) be a separating path. Then set \( u' = u(w_{m+1}, z_{m+2}) \) and \( v' = v(w_{m+1}, z_{m+2}) \). Also, let \( W' \) denote the black part of \( N' \). Proposition 9 implies that \( z_{m+1} \) is left of \( N' \). Propositions 8 and 16 then imply that all elements of \( B' \) are left of \( N' \).

**Claim.** All elements of \( A' \) are right of \( N' \).

**Proof.** Consider an element \( a \in A' \). Since \( a \) is left of \( N \), \( W \) cuts \( a \). Let \( p \) be the largest point of \( W \) that is also on \( aS y_0 \). Since \( z_{m+2} \) is left of \( N \), we know that \( W \) cuts \( z_{m+2} \). Let \( q \) be the least element of \( W \) that is also on \( x_0Tz_{m+2} \). Since the red and blue trees are disjoint, we know \( v \leq p < q \leq u \) in \( P \).

By the planarity of the drawing, we have that \( q \) is also the first point of \( x_0Tz_{m+2} \) that lies in \( W \). Also, the path \( qWP \) leaves \( x_0Tz_{m+2} \) from the right side.

Proposition 16 implies that there is a point \( r \in q Wu \) such that \( W \) cuts \( z_{m+1} \) at \( r \). In particular, we have \( q \leq r \leq z_{m+1} \) in \( P \).

In our research, we found it convenient to view the path \( M = x_0TqWpuS \) as a *pillar*. \( M \) is not a witnessing path, and it is not a separating path. Nevertheless, it has a useful property

\[
z \leq q \leq r \leq z_{m+1} \text{ in } P,
\]

for every \( z \) in \( M \).

In particular, this implies that \( W' \) does not intersect \( M \) as otherwise if \( w \) is an intersection point of \( W' \) and \( M \), we would \( w_{m+1} \leq w \leq z_{m+1} \) in \( P \), a clear contradiction.

Recall that \( W' \) hits the branch \( x_0Tz_{m+2} \) at element \( u' \). There are two options: either \( u' \) lies in the section \( x_0Tq \) (including \( q \)) or \( u' \) lies in the section \( qTz_{m+2} \) (excluding \( q \)). We want to exclude the first option. Indeed, if \( u' \) is on the path \( x_0Tq \), then

\[
w_{m+1} < u' \leq q \leq r \leq z_{m+1} \text{ in } P.
\]
which is a contradiction. We conclude that $u'$ is an element of $x_0 T z_{m+2}$ that occurs after $q$. This implies that the pillar $M$ leaves the path $x_0 T u'$ from the right side, as illustrated in Figure 10.

Now consider the red path $v' S y_0$. We observe that there is no point on $v' S y_0$ that also belongs to the pillar $M$. To see this, suppose that $w$ is a common point. Then

$$w_{m+1} \leq w \leq q \leq r < z_{m+2} \text{ in } P,$$

which is again clearly false.

To complete the proof, we simply recall that the pillar leaves the path $x_0 T u'$ from the right side and never touches $N'$ again. This means that $a$ must be right of $N'$, as desired. The final statement in the proof of the preceding claim is also illustrated in Figure 10.

We have now reached a contradiction since we have shown that $N'$ separates $A'$ and $B'$ with $|A'| = |B'| = m = 2h$, contradicting Proposition 17. This completes the proof of Proposition 18.

4.1. **Separating the Red and Blue Trees.** To illustrate the challenges we face in separating the blue and red trees, we show on the left side of Figure 11 how it can happen that $x_0 T b_i \cap y_0 S a_j$ can intersect for at least half the comparable pairs in $A_I \times B_I$. In this example, there is one "essential" crossing of the red and blue trees. On the right side of this figure, we show a small example with two essential crossings. With these examples in mind, it is conceivable that a more complex example might have arbitrarily many different essential crossings. Accordingly, it will take some effort to show that this cannot happen.
Now we begin the material necessary to separate the red and blue trees. Let \( Z \) be the non-empty subposet of \( P \) consisting of all elements of \( P \) that belong to a witnessing path from \( x_0 \) to \( y_0 \). If we restrict our drawing of \( G \) to the induced subgraph determined by the elements of \( Z \), we obtain a drawing without edge crossings of the cover graph of \( Z \). Furthermore, \( x_0 \) is the unique minimal element of \( Z \), and \( y_0 \) is the unique maximal element of \( Z \). We note that no element of \( A_I \cup B_I \) belongs to \( Z \).

Although we are working with a drawing of a cover graph, and not an order diagram, the fact that \( x_0 \) and \( y_0 \) are on the exterior face implies that when we restrict to the cover graph of \( Z \), and we take an element \( z \) of \( Z \), then like in an order diagram, the up covers of \( z \) must appear in a block as do the down covers of \( z \). Accordingly, when \( z \neq y_0 \), among the up covers of \( z \), there is a well defined left-to-right order (clockwise). And dually, when \( z \neq x_0 \), there is a left-to-right order on down covers of \( z \) (counterclockwise). See Figure 12.

It is easy to see that \( Z \) has dimension at most 2. We carry out a depth-first search of \( Z \), starting at \( x_0 \), with a local left-to-right preference, to obtain a linear extension \( L \) of \( Z \). Dually, we carry out a depth-first search of \( Z \) using a local right-to-left preference, to obtain a linear extension \( R \) of \( Z \). These two linear extensions form a realizer of \( Z \) as \( z < z' \) in \( Z \) if and only if \( z <_L z' \) and \( z <_R z' \). When \( u, v \in Z \) and \( u \parallel_P v \), we we will say that \( u \) is left of \( v \) (also \( v \) is right of \( u \)) when \( u <_L v \) and \( v <_R u \). Note that these two terms are transitive, e.g., if \( u \) is left of \( v \) and \( v \) is left of \( w \), then \( u \) is left of \( w \). We illustrate these concepts in Figure 12.

When \( W \) is a witnessing path from \( x_0 \) to \( y_0 \), then \( W \) is also a maximal chain in \( Z \). Therefore, if \( u \in Z \) and \( u \) is not on \( W \), then there is an element \( v \in W \) such that \( u \parallel_P v \).
Figure 12. The subposet $Z$ consists of all points of $P$ that are on witnessing paths from $x_0$ to $y_0$. The element 16 has two up covers 17, 21, and three down covers 10, 12, 15 (both lists sorted from left to right). The elements of $Z$ are labeled with the integers in $[25]$ according to the depth-first search linear extension $<_L$ that uses a local left-to-right preference. Note that 7 is left of 16 and 16 is left of 18. This implies 7 is left of 18. Also, 10 is right of 5.

We will say that $u$ is left of $W$ if $u$ is left of $v$. In a symmetric manner, we will say that a point $u \in Z$ that is not on $W$ is right of $W$ if there is a point $w$ on $W$ such that $u$ is right of $w$. It is easy to verify that when $u \in Z$ and $u$ is not on $W$, then either $u$ is left of $W$ or right of $W$, and these two options are mutually exclusive. Naturally, when $u \in Z$, the phrase $u$ is not left of $W$ means that either $u$ is on $W$ or $u$ is right of $W$.

The next two elementary propositions highlight the interplay of the terms left and right as applied to a pair of elements, and an element vs. a witnessing path.

**Proposition 19.** Let $W$ be a witnessing path from $x_0$ to $y_0$ and let $u$ and $v$ be a pair of points from $Z$ with $u \parallel_P v$. If $u$ is not right of $W$ and $v$ is not left of $W$, then $u$ is left of $v$.

**Proof.** Since $u$ is not right of $W$, we know that either $u$ is left of $W$ or $u$ is on $W$. In both cases, there exists $u'$ in $W$ such that $u \leq_L u'$. Analogously, since $v$ is not left of $W$, we know that either $v$ is right of $W$ or $v$ is on $W$. In both cases, there exists $v'$ in $W$ such that $v \leq_R v'$.

Since $u'$ and $v'$ both belong to $W$, they are comparable in $P$, i.e., $u' \leq v'$ or $v' \leq u'$ in $P$. In the first case, we conclude $u \leq_L u' \leq_L v' \leq_L v$ and since $u \parallel_P v$ in $P$, at least one of these inequalities must be strict. This proves that $u$ is left of $v$. In the second case, we conclude $v \leq_R v' \leq_R u' \leq_R u$, and since $u \parallel_P v$ in $P$, at least one of these inequalities must be strict. This proves that $v$ is right of $u$ (so $u$ is left of $v$). These observations complete the proof of the proposition. □
Proposition 20. Let $W$ be a witnessing path from $x_0$ to $y_0$, and let $u$ and $v$ be points of $Z$ with $u$ left of $v$. If $v$ is not right of $W$, then $u$ is left of $W$. Also, if $u$ is not left of $W$, then $v$ is right of $W$.

Proof. We prove the first assertion. The argument for the the second is symmetric. If $v$ is on $W$, then the fact that $u$ is left of $v$ implies $u$ is left of $W$. If $v$ is not on $W$, then our assumption forces that $v$ left of $W$. Choose $v'$ on $W$ such that $v$ is left of $v'$. Thus by transitivity, $u$ is left of $v'$ and so $u$ is left of $W$. □

Our fixed drawing of the cover graph of $Z$ splits the plane into regions: some number of bounded regions and one unbounded. We call such a bounded region a $Z$-face. Each element of $P$ that is not in $Z$ is in the interior of one of the regions. After adding two dummy element $z'$, $z''$ into $P$ (and $Z$) such that (1) $x_0 < z' < y_0$, $x_0 < z'' < y_0$ in $P$ and all these relations are covers; (2) $z'$ is the leftmost up (down) cover of $x_0$ ($y_0$); (3) $z''$ is the rightmost up (down) cover of $x_0$ ($y_0$); we can assume that any element of $P$ that is not in $Z$ is in the interior of one of the (bounded) $Z$-faces.

Each $Z$-face $F$ is bounded by two distinct witnessing paths that have only their starting and ending points in common. We let $x(F)$ denote the common starting point, and we let $y(F)$ denote the common ending point of these two witnessing paths. When we start at $x(F)$ and traverse the boundary of $F$ in a clockwise manner, we follow the left side of $F$ until we reach $y(F)$. Then we traverse the right side of $F$ backwards until we arrive back at $x(F)$.

When $F$ is a $Z$-face, no element $u$ of $P$ that is in the interior of $F$ satisfies $x(F) <_P u <_P y(F)$; otherwise this region would be split into smaller $Z$-faces. Also, a $Z$-face has no chords.

When $t, t'$ are elements of $Z$, we let $[t, t']$ consist of all elements $s \in Z$ with $t \leq s \leq t'$ in $Z$. However, we also consider $[t, t']$ as a region in the plane, i.e., we consider all points in the plane that are on witnessing paths from $t$ to $t'$ as well as all points in the plane that are in the interior of regions bounded by portions of two witnessing paths from $t$ to $t'$. Thus, there is a well defined left side and right side of $[t, t']$. Note that $[t, t']$ is a union of $Z$-faces and witnessing paths. It might be that the left side and the right side of $[t, t']$ can share points and even edges of witnessing paths. We illustrate these concepts in Figure 13.

When $F$ is a $Z$-face, we define the left side path of $F$ by concatenating the following three paths: (1) the left side of $[x_0, x(F)]$; (2) the left side of $F$; and (3) the left side of $[y(F), y_0]$. The right side path of $F$ is defined symmetrically.

Let $F$ and $F'$ be two distinct $Z$-faces. We say that $F$ is under $F'$ if $y(F) \leq_P x(F')$. Dually, we say that $F$ is over $F'$ if $x(F) \geq_P y(F')$. We say that $F$ is left of $F'$ if no point on the boundary of $F$ is right of the left side path of $F'$. Symmetrically, we say that $F$ is right of $F'$ if no point on the boundary of $F$ is left of the right side path of $F'$. Proposition 20 implies that when $F$ is left of $F'$, there is a point $u$ on the boundary of
Figure 13. An interval $[t, t']$ with its two sides bolded.

Figure 14. The left side path and the right side path for the face 1 are shown using thick lines. Shadow face 1 is over 2, 3 and 4. It is under 5 and 6. All other Z-faces are either left or right of 1. In particular, 7, 8, 9 are left of 1, while 10, 11, 12 are right of 1.

$\mathcal{F}$ that is left of the left side path of $\mathcal{F}'$. Symmetrically, when $\mathcal{F}$ is right of $\mathcal{F}'$, there is a point $v$ on the boundary of $\mathcal{F}$ that is right of the right side path of $\mathcal{F}'$. These observations give a formal argument for the natural conclusion that if $\mathcal{F}$ and $\mathcal{F}'$ are distinct Z-faces, then $\mathcal{F}$ is either over, under, left of, or right of $\mathcal{F}'$. Furthermore, the four options are mutually exclusive. We illustrate the concept of over, under, left and right for Z-faces in Figure 14.

When $u \in P$ and $u$ is not in $Z$, there is a unique Z-face $\mathcal{F}_u$ containing $u$ in its interior. We let $y_u = y(\mathcal{F}_u)$ and $x_u = x(\mathcal{F}_u)$. We note that when $(a, b) \in I$, then $a \not\in Z$ and $b \not\in Z$. A witnessing path from $a$ to $y_0$ has to leave the interior of $\mathcal{F}_a$, and this implies $a < y_a$ in
A pair \((a,b)\) \(\in I\) will be called a *same-face pair* if \(\mathcal{F}_a = \mathcal{F}_b\); otherwise we will say that \((a,b)\) is a *diff-face pair*. When \((a,b)\) is a diff-face pair, then \(\mathcal{F}_a\) is not under \(\mathcal{F}_b\), as this would imply \(a < y_a \leq x_b < b\) in \(P\). It follows that \(\mathcal{F}_a\) is either over, left of or right of \(\mathcal{F}_b\).

**Lemma 21.** The set of all diff-face pairs in \(I\) can be covered by two reversible sets.

**Proof.** Let \(M_1\) be the set consisting of all diff-face pairs \((a,b)\) \(\in I\) such that either \(\mathcal{F}_a\) is left of \(\mathcal{F}_b\) or \(\mathcal{F}_a\) is over \(\mathcal{F}_b\). Let \(M_2\) be the set consisting of all diff-face pairs \((a,b)\) \(\in I\) such that either \(\mathcal{F}_a\) is right of \(\mathcal{F}_b\) or \(\mathcal{F}_a\) is over \(\mathcal{F}_b\). Clearly, the two sets \(M_1\) and \(M_2\) cover the set of all diff-face pairs in \(I\). We now show that \(M_2\) is reversible. The argument to show that \(M_1\) is reversible is symmetric. Suppose to the contrary that \(((a_1,b_1),\ldots,(a_k,b_k))\) is an alternating cycle of diff-face pairs from \(M_2\). For each \(i \in [k]\), let \(z_i\) be the \(<_L\)-largest point on the boundary of \(\mathcal{F}_{b_i}\) with \(z_i <_P b_i\). There is such an element since \(x_{b_i} <_P b_i\).

**Claim.** \(z_i <_L z_{i+1}\) for all \(i \in [k]\) (cyclically).

**Proof.** Let \(i \in [k]\). Since \((a_i,b_i)\) is a diff-face pair, \(\mathcal{F}_{a_i}\) and \(\mathcal{F}_{b_i}\) are distinct \(Z\)-faces. We know that \(a_i <_P b_{i+1}\). Let \(W_i\) be a witnessing path from \(a_i\) to \(b_{i+1}\). Then let \(u_i\) be the first point of \(W_i\) that is on the boundary of \(\mathcal{F}_{a_i}\), and let \(v_{i+1}\) be the last point of \(W_i\) that is on the boundary of \(\mathcal{F}_{b_{i+1}}\). Then \(u_i <_P v_{i+1}\), and \(v_{i+1} <_L z_{i+1}\). Since \(<_L\) is a linear extension of \(P\), this implies \(u_i <_L z_{i+1}\).

If \(u_i \leq_<_P z_i\), then \(a_i <_P u_i \leq z_i <_P b_i\) so \(a_i <_P b_i\), which is false. We conclude that \(u_i \not<_P z_i\). If \(z_i <_P u_i\), then we have \(z_i <_L u_i \leq_<_L z_{i+1}\). This implies \(z_i <_L z_{i+1}\) as desired.

So we may assume that \(u_i \not<_L z_i\) and \(z_i \not<_L u_i\) in \(P\). In other words, \(u_i \parallel z_i\) in \(P\). Since \((a_i,b_i)\) \(\in M_2\), either \(\mathcal{F}_{a_i}\) is over \(\mathcal{F}_{b_i}\) or \(\mathcal{F}_{a_i}\) is right of \(\mathcal{F}_{b_i}\). If \(\mathcal{F}_{a_i}\) is over the face \(\mathcal{F}_{b_i}\), then we have \(u_i \leq y_{a_i} \leq x_{b_i} < z_i\) in \(P\) but we are now assuming that \(u_i \parallel z_i\), so this option cannot hold. We conclude that \(\mathcal{F}_{a_i}\) is right of \(\mathcal{F}_{b_i}\). Let \(W\) be the left side path of \(\mathcal{F}_{a_i}\). Then no point on the boundary of \(\mathcal{F}_{b_i}\) is right of \(W\). In particular, \(z_i\) is not right of \(W\). On the other hand, the point \(u_i\) is on the boundary of \(\mathcal{F}_{a_i}\). Therefore, \(u_i\) is not left of \(W\). Now Proposition 19 implies \(z_i\) is left of \(u_i\). Altogether we have \(z_i < u_i \leq z_{i+1}\) in \(L\). This completes the proof of the claim.

To complete the proof of the lemma, we simply note that the statement of the claim cannot hold for all \(i \in [k]\) cyclically.

**Lemma 22.** For every strict alternating cycle \(((a_1,b_1),\ldots,(a_k,b_k))\) of same-face pairs in \(I\), there is a \(Z\)-face \(\mathcal{F}\) such that all elements \(a_1,\ldots,a_k,b_1,\ldots,b_k\) are in the interior of \(\mathcal{F}\).
Proof. We assume to the contrary that \(((a_1, b_1), \ldots, (a_k, b_k))\) is a strict alternating cycle of same-face pairs from \(I\), and there is no \(Z\)-face that contains all elements of the cycle in its interior. Of all such cycles, we assume further that \(k\) is minimum.

Claim 1. There do not exist distinct integers \(i, j \in [k]\) such that the pairs \((a_i, b_i)\) and \((a_j, b_j)\) are in the same \(Z\)-face.

Proof. Suppose that for some \(i \neq j\) all four elements involved in \((a_i, b_i), (a_j, b_j)\) lie in the same \(Z\)-face. Since our alternating cycle is a counterexample, we do not have all the pairs lying in the same \(Z\)-face, so we know that \(k \geq 3\). After a relabeling, we may assume that \(j = k\) and \(2 \leq i \leq k - 1\). However, this implies that

\[ ((a_1, b_1), \ldots, (a_{i-1}, b_{i-1}), (a_i, b_i)) \]

is an alternating cycle of same-face pairs from \(I\). This is a contradiction unless all the pairs on this cycle belong the same \(Z\)-face. In this case, we consider the alternating cycle

\[ ((a_i, b_i), (a_{i+1}, b_{i+1}), \ldots, (a_k, b_k)). \]

Again, we have a strict alternating cycle. However, now it is clear that not all the pairs on this cycle belong to the same \(Z\)-face. Furthermore, the length of this cycle is less than \(k\). The contradiction completes the proof of the claim.

For each \(i \in [k]\), let \(F_i\) be the common \(Z\)-face \(F_{a_i} = F_{b_i}\), let \(x_i = x_{b_i}\), and let \(y_i = y_{a_i}\). Let \(W_i = W_i(a_i, b_{i+1})\) be a witnessing path. Then let \(u_i\) be the lowest point of \(W_i\) that is on the boundary of \(F_i\), and let \(v_{i+1}\) be the highest point of \(W_i\) that is on the boundary of \(F_{i+1}\). We note that \(a_i < v_i < v_{i+1} < b_{i+1}\).

Claim 2. For all \(i, j \in [k]\), \(u_i \leq v_j\) if and only if \(j = i + 1\) (cyclically).

Proof. We already know that \(u_i \leq v_{i+1}\) for all \(i \in [k]\). Now suppose \(j \neq i + 1\) and \(u_i \leq v_j\). Then \(a_i < u_i \leq v_j < b_j\). This implies \(a_i < b_j\). Now we have contradicted the assumption that our original cycle is strict. With this observation, the proof of the claim is complete.

Claim 2 implies that \(((u_1, v_1), \ldots, (u_k, v_k))\) is a strict alternating cycle of incomparable pairs in \(Z\). Let \(i \in [k]\). Since \(a_i \parallel v_i\), and both \(u_i\) and \(v_i\) are on the boundary of \(F_i\), it implies that they are on opposite sides of \(F_i\). Also, \(\{u_i, v_i\} \cap \{x_i, y_i\} = \emptyset\). Furthermore, the statement \(u_i\) is left of \(v_i\) means the same as saying \(u_i\) is on the left side of \(F_i\) and \(v_i\) is on the right side of \(F_i\). A symmetric statement holds when \(u_i\) is right of \(v_i\).

Claim 3. For each \(i \in [k]\), the following statements hold.

(i) If \(u_i < v_i\), then \(u_{i+1} < v_{i+1}\) and \(u_{i+1} < u_i\).

(ii) If \(u_i < v_i\), then \(u_{i+1} < v_{i+1}\) and \(u_{i+1} < u_i\).
Figure 15. The point $v_{i+1}$ must be on the left side of $[u_i, y_0]$. This forces $v_{i+1}$ to be on the right side of $F_{i+1}$. In turn, this forces $u_{i+1}$ to be on the left side of $F_{i+1}$.

Proof. We prove the first statement. The proof of the second is symmetric. Let $i \in [k]$. Then $u_i$ is on the left side of $F_i$, and $v_i$ is on the right side of $F_i$.

Since $u_{i+1}$ and $v_{i+1}$ lie on the boundary of $F_{i+1}$, and $u_i \preceq_P v_{i+1}, u_i \parallel_P u_{i+1}$, we conclude that $v_{i+1}$ must be on the boundary (left side or right side) of $[u_i, y_0]$ while the point $u_{i+1}$ as well as all points in plane that are in the interior of $F_{i+1}$ are in the exterior of the interval $[u_i, y_0]$.

First, we assume that $v_{i+1}$ is on the right side of $[u_i, y_0]$. Clearly, the right side of this interval is the portion of the left side of $F_i$ from $u_i$ to $y_i$ concatenated with the right side of the interval $[y_i, y_0]$. If $v_{i+1}$ lies on the left side of $F_i$ (somewhere starting from $u_i$ but before $y_i$), then since $F_i$ and $F_{i+1}$ are distinct faces, this would force $v_{i+1}$ to be on the right side of $F_{i+1}$. In turn, this would imply that $u_{i+1}$ is on the left side of $F_{i+1}$ and therefore $u_{i+1} <_L v_{i+1}$. Also when we extend any witnessing path from $u_i$ to $v_{i+1}$ to a witnessing path $W$ from $x_0$ to $y_0$, we clearly have $u_i$ on $W$ and $u_{i+1}$ left of $W$. Now by Proposition 19, we conclude that $u_{i+1}$ is left of $u_i$ as desired. If $v_{i+1}$ lies on the right side of $[y_i, y_0]$, then $v_i < y_i \leq v_{i+1}$ in $P$, which cannot hold in our strict alternating cycle.

It remains only to consider the case that $v_{i+1}$ is on the left side of the interval $[u_i, y_0]$. This forces $v_{i+1}$ to be on the right side of $F_{i+1}$ and therefore $u_{i+1}$ is on the left side of $F_{i+1}$. Thus we get $u_{i+1} <_L v_{i+1}$. This is the situation illustrated in Figure 15. Again when we extend any witnessing path from $u_i$ to $v_{i+1}$ to a witnessing path $W$ from $x_0$ to $y_0$, we clearly have $u_i$ on $W$, and $u_{i+1}$ left of $W$. Now by Proposition 19, we conclude that $u_{i+1}$ is left of $u_i$ as desired. With this observation, the proof of the claim is complete. □

To complete the proof of the lemma, we simply note that the statement of the claim cannot hold for all $i \in [k]$ cyclically. □
When \( \mathcal{F} \) is a \( Z \)-face, let \( I(\mathcal{F}) \) consist of those pairs \((a, b)\) ∈ \( I \) such that \( \mathcal{F} = \mathcal{F}_a = \mathcal{F}_b \). Then Lemma 21 and Lemma 22 imply

\[
\rho(I) \leq 2 + \max_{\mathcal{F}} \rho(I(\mathcal{F})).
\]

With this material as background, we may assume that:

(i) The boundary of \( G \) is the boundary of a \( Z \)-face \( \mathcal{F} \).

(ii) \( I = I(\mathcal{F}) \), \( x_0 = x(\mathcal{F}) \), and \( y_0 = y(\mathcal{F}) \).

(iii) If \((a, b)\) ∈ \( I \), then \( x_0 \leq_P b \) and \( a <_P y_0 \).

Let \(((a_1, b_1), \ldots, (a_n, b_n))\) be a directed path in \( I \). Suppose that \((i, j)\) is a pair of distinct integers in \([n]\) such that \( x_0Tb_j \) intersects \( y_0Sa_i \). Let \( u = u(a_i, b_j) \) and \( v = v(a_i, b_j) \) be, respectively the least and greatest element of \( P \) common to \( x_0Tb_j \) and \( y_0Sa_i \). Then \( x_0 \leq_P u \leq_P v \leq_P y_0 \). Since \( \mathcal{F} \) is a \( Z \)-face, no point in the interval \([u, v]\) is in the interior of \( \mathcal{F} \). It follows that \([u, v]\) is a portion of one of the two sides of \( \mathcal{F} \). Furthermore, all points and edges of \( x_0Tb_j \) after \( v \) are in the interior of \( \mathcal{F} \), and all points and edges of \( a_iSy_0 \) before \( u \) are in the interior of \( \mathcal{F} \).

Now suppose that \( i < j \). It is easy to see that the following three statements hold:

\begin{align*}
S(1): & \quad x_0Tv \text{ and } uSy_0 \text{ are portions of the right side of } \mathcal{F}. \\
S(2): & \quad I \leq i' < i, \text{ then } uSy_0 \text{ is a terminal portion of } a_iSy_0. \\
S(3): & \quad j < j' \leq n, \text{ then } x_0Tv \text{ is an initial portion of } x_0Tb_j'.
\end{align*}

If \( i > j \), then there is a symmetric set of three statements for the left side of \( \mathcal{F} \).

The proof of the following proposition completes the proof of our Theorem.

**Proposition 23.** If \(((a_1, b_1), \ldots, (a_n, b_n))\) is a directed path in \( H_I \), then \( n \leq 34h + 9 \).

**Proof.** We argue by contradiction and assume that there is a directed path \(((a_1, b_1), \ldots, (a_n, b_n))\) in \( H_I \) with \( n = 34h + 10 \). Set \( s = 6h + 2 \), and note that \( n = 5s + 4h \).

If \( X \) is any subset of \([n]\) with \(|X| = s \), then Proposition 18 implies that there are distinct integers \( i, j \in X \) such that \( x_0Tb_j \) intersects \( a_iSy_0 \). We apply this observation to the set \( X = [s] \). We give the balance of the argument under the assumption that \( i < j \). From the details of the argument, it will be clear that the proof when \( i > j \) is symmetric.

Since \( 1 \leq i < j \leq s \), we know that statements \( S(1), S(2), S(3) \) hold. For each \( \alpha \) with \( i \leq \alpha \leq n \), let \( u_\alpha \) be the lowest point on the right side of \( \mathcal{F} \) that belongs to \( a_\alphaSy_0 \). If \( \alpha < n \), then \( u_\alpha \leq_P u_{\alpha+1} \). For each \( \beta \) with \( j \leq \beta \leq n \), let \( v_\beta \) be the highest point on the right side of \( \mathcal{F} \) that belongs to \( x_0Tb_\beta \). Since \( a_\beta \parallel_P b_\beta \), we know \( v_\beta <_P u_\beta \). Furthermore, if \( \beta < n \), then \( v_\beta \leq_P v_{\beta+1} \).
Figure 16. On the left, we show an intersection between the red and blue trees in the $Z$-face $F$. On the right, we show a forced intersection, and the resulting contradiction completes the proof.

An important consequence of the previous paragraph is that for every $\beta$ with $j \leq \beta < n$, the path $x_0Tb_\beta$ contains a non-trivial portion of the right side of $F$. Since we are working with a single $Z$-face $x_0Tb_\beta$ cannot hit both the left side and the right side of $F$. Therefore, $x_0Tb_\beta$ stays disjoint from the left side of $F$. This implies that whenever $x_0Tb_\beta$ intersects $a_\alpha Sy_0$, we have $\alpha < \beta$.

Claim 1. If $j \leq k \leq n - s$, then $u_k \leq_P v_{k+s}$.

Proof. Consider the set $X = \{k, k+1, \ldots, k+s\}$. Since this set has size $s + 1$, by Proposition 18 there is some distinct pair $\alpha, \beta$ of elements of this set such that $x_0Tb_\beta$ intersects $a_\alpha Sy_0$. Our remarks just above require $\alpha < \beta$, and this implies $u_\alpha \leq v_\beta$. We conclude that $u_k \leq u_\alpha \leq v_\beta \leq v_{k+s}$ in $P$. \hfill $\Box$

We note that if $j \leq k \leq m \leq n$, then $v_j \prec_P u_j \leq_P u_m$, so $v_j \prec_P u_m$. As illustrated on the left side of Figure 16, we then have the following comprehensive inequality:

$$v_s < u_s \leq v_{2s} < u_{2s+2h} \leq v_{3s+2h} < u_{3s+2h} \leq v_{4s+2h} < u_{4s+4h} \leq v_{5s+4h} < u_{5s+4h} \text{ in } P.$$ 

Let $N_1 = N(a_{3s+2h}, b_s)$ be a separating path. We note that $x_0Tb_s$ and $a_{3s+2h}Sy_0$ are disjoint. Let $W_1 = W(w_1, z_1)$ be the (necessarily non-trivial) witnessing path that forms the black part of $N_1$. Proposition 9 implies that $b_{3s+2h}$ is right of $N_1$. 
Referring to the right side of Figure 16, let $R$ be the region in the plane formed by $v, Tz_1W_1w_1Su_{3s+2h}$ and the portion of the right side of $F$ between $v$ and $u_{3s+2h}$. Clearly, when $u$ is an element of $A_1 \cup B_1$, we have $u$ is right of $N_1$ if and only if $u$ is in the interior of $\mathcal{R}$.

We assert that there is no point $v$ in the blue tree with $v \geq_P u_s$ such that $v \in W_1$. To see this, the existence of $v$ would imply:

$$a_s <_P u_s \leq_P v \leq_z b_s.$$  

This would imply $a_s <_P b_s$, which is false. Therefore, our assertion holds. From this, it follows that all points of $B_2 = \{b_{4s+2h}, \ldots, b_{4s+4h}\}$ are left of $N_1$.

**Claim 2.** $W_1$ does not intersect $a_{5s+4h}Sy_0$.

**Proof.** If the claim fails, then Proposition 16 implies that $W_1$ intersects $aSy_0$ for every $a \in A_2 = \{a_{4s+2h}, \ldots, a_{4s+4h}\}$. We assert that in fact, $a$ is right of $N_1$ for every $a \in A_2$. If this assertion does not hold, then it is easy to see that $a$ is enclosed by $a_{3s+2h}$. Therefore, our assertion holds. However, this now implies that $N_1$ separates $A_2$ from $B_2$. This is a contradiction with Proposition 17 since these two sets have size larger than $2h - 1$. □

As a consequence of Claim 2, we know that $a_{5s+4h}$ is left of $N_1$. Now let $N_2 = N(a_{5s+4h}, b_{3s+2h})$ be a separating path, and let $W_2 = W(w_2, z_2)$ be the non-trivial witnessing path that forms the black part of $N_2$. Using symmetric arguments, the following statements hold: (1) $b_{5s+4h}$ is right of $N_2$; (2) there is no point $w$ of the red tree with $w \leq_P v_{5s+4h}$ such that $w \in W_2$; (3) all elements of $A_2$ are right of $N_2$; and (4) $W_2$ does not intersect $x_0Tb_s$.

Since $a_{5s+4h}$ is left of $N_1$, and $b_{3s+2h}$ is right of $N_1$, the witnessing path $W = a_{5s+4h}Sw_2W_2z_2Tb_{3s+2h}$ must intersect the boundary of $\mathcal{R}$. Clearly, this requires that there is a point $v$ common to $W_2$ and $W_1$. This implies

$$a_{3s+2h} \leq_P w_1 \leq_P v \leq_P z_2 \leq_P b_{3s+2h}.$$  

In turn, this implies $a_{3s+2h} <_P b_{3s+2h}$, which is false. The contradiction completes the proof of the proposition. □

And as noted previously, this completes the proof of Lemma 15, as well as the principal theorem of the paper.

### 5. Closing Comments

Since we have not been able to disprove that $\dim(P) = \mathcal{O}(h)$ we comment that our proof that for $\mathcal{O}(h^6)$ has three steps where improvements might be possible. Do we really need the $\mathcal{O}(h^3)$ factor in the transition from singly constrained to doubly constrained set of incomparable pairs? When $I$ is a set of doubly constrained pairs, did we need another
factor of $h$ to transition to the doubly exposed case? Could $\dim(I)$ be linear in $\rho(I)$ when $I$ is doubly exposed in $P$?

Although we believe the establishment of a polynomial bound for dimension in terms of height for posets with planar cover graphs is intrinsically interesting, we find the results of Section 3, where height plays no role, particularly intriguing. Indeed, we hope that insights from this line of research may help to resolve the following long-standing conjecture.

**Conjecture 24.** For every $n \geq 2$, there is a least positive integer $d$ so that if $P$ is a poset with a planar cover graph and $\dim(P) \geq d$, then $P$ contains the standard example $S_n$.

Apparently, the first reference in print to Conjecture 24 is in an informal comment on page 119 of [13], published in 1991. However, the problem goes back at least 10 years earlier. In 1978, Trotter [12] showed that there are posets that have large dimension and have planar cover graphs. In 1981, Kelly [8] showed that there are posets that have large dimension and have planar order diagrams. In both of these constructions, the fact that the posets have large dimension is evidenced by large standard examples that they contain. The belief that large standard examples are necessary for large dimension among posets with planar cover graphs grew naturally from these observations.

To attack Conjecture 24, it is tempting to believe that we can achieve a transition from a singly constrained poset to a doubly exposed poset, independent of height, by allowing a considerable reduction in the dimension $d$.

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**References**

POSETS WITH PLANAR COVER GRAPHS