# BOOLEAN DIMENSION AND DIM-BOUNDEDNESS: PLANAR COVER GRAPH WITH A ZERO 

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#### Abstract

In 1989, Nešetřil and Pudlák posed the following challenging question: Do planar posets have bounded Boolean dimension? We show that every poset with a planar cover graph and a unique minimal element has Boolean dimension at most 13. As a consequence, we are able to show that there is a reachability labeling scheme with labels consisting of $\mathcal{O}(\log n)$ bits for planar digraphs with a single source. The best known scheme for general planar digraphs uses labels with $\mathcal{O}\left(\log ^{2} n\right)$ bits [Thorup JACM 2004], and it remains open to determine whether a scheme using labels with $\mathcal{O}(\log n)$ bits exists. The Boolean dimension result is proved in tandem with a second result showing that the dimension of a poset with a planar cover graph and a unique minimal element is bounded by a linear function of its standard example number. However, one of the major challenges in dimension theory is to determine whether dimension is bounded in terms of standard example number for all posets with planar cover graphs.


## 1. Introduction

1.1. Dimension and Boolean dimension. Partially ordered sets, called posets for short, are combinatorial structures with applications in many areas of mathematics and theoretical computer science. The most widely studied measure of a poset's complexity is its dimension, as defined by Dushnik and Miller [5]. A linear extension $L$ of a poset $P$ is a total order on the elements of $P$ such that if $x \leqslant y$ in $P$, then $x \leqslant y$ in $L$. A realizer of a poset $P$ is a set $\left\{L_{1}, \ldots, L_{d}\right\}$ of linear extensions of $P$ such that

$$
x \leqslant y \Longleftrightarrow\left(x \leqslant y \text { in } L_{1}\right) \wedge \cdots \wedge\left(x \leqslant y \text { in } L_{d}\right)
$$

for all $x, y \in P$. The dimension, denoted by $\operatorname{dim}(P)$, is the minimum size of a realizer of $P$.
Realizers provide a compact scheme for handling comparability queries: Given a realizer $\left\{L_{1}, \ldots, L_{d}\right\}$ for a poset $P$, then a query of the form "is $x \leqslant y$ ?" can be answered by looking at the relative position of $x$ and $y$ in each of the $d$ linear extensions of the realizer.

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Figure 1. The standard example $S_{6}$ (left). Kelly's planar poset containing $S_{6}$ as a subposet (right). The wheel construction of the poset with a planar cover graph, with a single minimal element and a single maximal element, and containing $S_{6}$ as a subposet (bottom). Note also that at the bottom we have a planar drawing of the cover graph (not diagram) and the poset relation goes "inwards".

As noted by Gambosi, Nešetřil and Talamo [7], these observations motivate a more general encoding of posets. A Boolean realizer of a poset $P$ is a sequence of linear orders, i.e. not necessarily linear extensions, $\left(L_{1}, \ldots, L_{d}\right)$ of the elements of $P$ and a $d$-ary Boolean formula $\phi$ such that

$$
x \leqslant y \Longleftrightarrow \phi\left(\left(x \leqslant y \text { in } L_{1}\right), \ldots,\left(x \leqslant y \text { in } L_{d}\right)\right)=1,
$$

for every $x, y \in P$. The Boolean dimension of $P$, denoted by $\operatorname{bdim}(P)$, is a minimum size of a Boolean realizer. Clearly, for every poset $P$ we have $\operatorname{bdim}(P) \leqslant \operatorname{dim}(P)$.

As is well known, when $n \geqslant 4$, the maximum dimension of a poset on $n$ elements is $\lfloor n / 2\rfloor$. The upper bound in this statement is evidenced by the following construction: When $d \geqslant 2$, the standard example $S_{d}$ is a poset whose ground set is $\left\{a_{1}, \ldots, a_{d}, b_{1}, \ldots, b_{d}\right\}$ with $a_{i}<b_{j}$ in $S_{d}$ if and only if $i \neq j$ (see Figure 1). Dushnik and Miller [5] observed that $\operatorname{dim}\left(S_{d}\right)=d$. On the other hand, it is an instructive exercise to show that $\operatorname{bdim}\left(S_{d}\right) \leqslant 4$ for every $d \geqslant 2$. In [12], Nešetřil and Pudlák showed that the maximum Boolean dimension among posets on $n$ elements is $\Theta(\log n)$.

The cover graph of a poset $P$ is the graph whose vertices are the elements of $P$ with edge set $\{x y \mid x<y$ in $P$ and there is no $z$ with $x<z<y$ in $P\}$. Somewhat unexpectedly, posets
with planar cover graphs can have arbitrarily large dimension. In 1978, Trotter [15] gave the "wheel construction" illustrated in Figure 1. This construction shows that there are posets with planar cover graphs, a unique minimal element, a unique maximal element, and arbitrarily large dimension. In 1981, Kelly [9] gave a construction (also illustrated in Figure 1) that shows that there are posets with planar order diagrams that have arbitrarily large dimension. In both of these constructions, the fact that dimension is large stems from the fact that the posets contain large standard examples. Note that the posets in Kelly's construction have a large number of minimal elements. However, their cover graphs have pathwidth at most 3. On the other hand, the posets in Trotter's construction have a unique minimal element, but their cover graphs have unbounded treewidth. We also note that the Boolean dimension of the posets in both constructions is bounded.

Here is an intriguing question posed in [12] that remains unanswered to this day:
Problem A (Nešetřil, Pudlák 1989). Do posets that have planar cover graphs ${ }^{1}$ have bounded Boolean dimension?

Nešetřil and Pudlák suggested an approach for a negative resolution of this question that involves an auxiliary Ramsey-type problem for planar posets. However, no progress in this direction has been made. From the positive side, researchers have in recent years investigated conditions on cover graphs that bound Boolean dimension. Mészáros, Micek and Trotter [11] proved that the Boolean dimension of a poset is bounded in terms of the Boolean dimension of its 2-connected blocks. Felsner, Mészáros, and Micek [6] proved that posets with cover graphs of bounded treewidth have bounded Boolean dimension. On the other hand, as noted previously, the Kelly examples have unbounded dimension, but their cover graphs have pathwidth at most 3.

The first of the two principal theorems in this paper is:
Theorem 1. If $P$ is a poset with a planar cover graph and a unique minimal element, then

$$
\operatorname{bdim}(P) \leqslant 13
$$

1.2. Dim-boundedness. As referenced in the abstract, our proof of Theorem 1 is proved in tandem with a second result that is relevant to a conjecture that is even older than Problem A. The standard example number of a poset $P$, denoted se $(P)$, is set to be 1 if $P$ does not contain a subposet isomorphic to the standard example $S_{2}$; otherwise se $(P)$ is the largest $d \geqslant 2$ such that $P$ contains a subposet isomorphic to the standard example $S_{d}$. Obviously, a poset that contains a large standard example has large dimension, i.e., $\operatorname{dim}(P) \geqslant \operatorname{se}(P)$ for every poset $P$. As is well known, if $P$ is a distributive lattice, and $\operatorname{dim}(P) \geqslant 3$, then $\operatorname{dim}(P)=\operatorname{se}(P)$, so there are important classes of posets for which the inequality $\operatorname{dim}(P) \geqslant \operatorname{se}(P)$ is tight.

However, dimension can be large without the presence of standard examples. The class of posets with standard example number 1 is the class of interval orders, and this special class of posets has been studied extensively in the literature. As noted in [3], the maximum dimension of an interval order on $n$ elements is known to be within $\mathcal{O}(1)$ of $\log \log n+\frac{1}{2} \log \log \log n$.

More generally, for a fixed value of the standard example number, the maximum value of dimension is polynomial in the size of the poset. In 2020, Biró, Hamburger, Kierstead, Pór,

[^1]Trotter and Wang [1] used random methods for bipartite posets to show that there is a constant $n_{0}$ such that if $n>n_{0}$, then there is an $n$-element poset $P$ with $\operatorname{se}(P)=2$ and $\operatorname{dim}(P)>\frac{n^{1 / 6}}{8 \log n}$. Moreover, for every integer $d \geqslant 3$ there is a constant $\alpha_{d}>0$ so that $\lim _{d \rightarrow \infty} \alpha_{d}=1$, and the maximum dimension of $n$-element posets $P$ with se $(P)<d$ is $\Omega\left(n^{\alpha_{d}}\right)$. Upper bounds are more challenging. In 2015, Biró, Hamburger and Pór [2] proved that for a fixed value of $d \geqslant 3$, the maximum dimension among posets on $n$ elements that have standard example number less than $d$ is $o(n)$. It is a great challenge to verify if $o(n)$ can be replaced by $\mathcal{O}\left(n^{\alpha}\right)$ with some $\alpha<1$ depending only on $d$.

A class $\mathcal{C}$ of posets is dim-bounded if there is a function $f: \mathbb{N} \rightarrow \mathbb{N}$ such that $\operatorname{dim}(P) \leqslant f(\operatorname{se}(P))$ for every poset $P$ in $\mathcal{C}$. There is an analogous research theme in graph theory: When $G$ is a graph, $\chi(G)$ denotes the chromatic number of $G$, and $\omega(G)$ denotes the clique number of $G$. In this setting, we have the trivial inequality: $\chi(G) \geqslant \omega(G)$. As is well known, there are triangle-free graphs with arbitrarily large chromatic number. Regardless, researchers have found interesting classes of graphs where chromatic number is bounded in terms of clique number. Such classes are called $\chi$-bounded. A recent survey by Scott and Seymour [13] has just appeared, and this paper gives an excellent summary of the substantial progress on $\chi$ boundedness achieved within the last decade.

For posets, we have the following long standing conjectures:

## Conjecture B.

(i) The class of posets that have planar diagrams is dim-bounded.
(ii) The class of posets that have planar cover graphs is dim-bounded.

We believe the first published reference to Conjecture B.(i) is an informal comment on page 119 in [16] published in 1992. However, the conjecture was circulating among researchers soon after the constructions illlustrated in Figure 1 appeared. Accordingly, Conjecture B.(i) is more than 40 years old and obviously Conjecture B.(ii) is a stronger statement.

Here is our second principal theorem.
Theorem 2. The class of posets with a planar cover graph and a unique minimal element is dim-bounded. Specifically, given a poset $P$ in this class, we have

$$
\operatorname{dim}(P) \leqslant 2 \operatorname{se}(P)+2 .
$$

Since $\operatorname{dim}(P) \geqslant \operatorname{se}(P)$ for all $P$, the upper bound in Theorem 2 is best possible to within a multiplicative factor of 2 .

The remainder of the paper is organized as follows. In the next section, we give a brief treatment of reachability labeling schemes and the connection with Boolean dimension. This is followed by a section with essential preparatory material on posets, dimension and Boolean dimension. The arguments for each of our two main theorems can be split into two parts. The first part is the same for both theorems, and this common part will be contents of Section 4. This will be followed by sections giving the second parts of the proofs for each of the two main theorems, with the result for Boolean dimension given first.

## 2. Reachability labeling schemes and Boolean dimension

Given two vertices $u, v$ in a directed graph (digraph, for short), we say that $u$ can reach $v$ if there is a directed path from $u$ to $v$ in the digraph. A class of digraphs $\mathcal{C}$ admits an $f(n)$ bit reachability scheme if there exists a function $A:\left(\{0,1\}^{*}\right)^{2} \rightarrow\{0,1\}$ such that for every positive integer $n$ and every $n$-vertex digraph $G \in \mathcal{C}$ there exists $\ell: V(G) \rightarrow\{0,1\}^{*}$ such that $|\ell(v)| \leqslant f(n)$ for each vertex $v$ of $G$, and such that, for every two vertices $u, v$ of $G$

$$
A(\ell(u), \ell(v))= \begin{cases}1 & \text { if } u \text { can reach } v \text { in } G \\ 0 & \text { otherwise }\end{cases}
$$

Bonamy, Esperet, Groenland, and Scott [4] have recently devised a reachability labeling scheme for the class of all digraphs using labels of length at most $n / 4+o(n)$. This is best possible up to the lower order term as a simple counting argument forces every reachability labeling scheme for the class of all $n$-vertex digraphs to use a label of length at least $n / 4-o(n)$.

Quoting from [8] "In this paper we focus on the planar case, which feels particularly relevant when you live on a sphere." In 2004, Thorup [14] presented an $\mathcal{O}\left(\log ^{2} n\right)$-bit reachability labeling scheme for planar digraphs. It remains open to answer whether more efficient schemes exist.

Problem C. Do planar digraphs admit an $\mathcal{O}(\log n)$-bit reachability labeling scheme?
Planar digraphs with a single source $s$ and a single sink $t$ admitting a plane drawing with both $s$ and $t$ on the exterior face are called st-planar graphs. These graphs have a very simple structure, and within this class, it is straightforward to construct a $\mathcal{O}(\log n)$-bit reachability labeling scheme. This observation was an important tool in related work on reachability oracles by Holm, Rotenberg, and Thorup [8].

There is a standard technique for reducing reachability queries in digraphs to comparability queries in posets: Given a digraph $G$, contract each strongly connected component of $G$ to a single vertex. Let $G^{\prime}$ be the resulting digraph which is obviously acyclic. Note also that if $G$ is planar, then $G^{\prime}$ is planar as well. Now given a labeling of $G^{\prime}$ we extend it to a labeling of $G$ by assigning to each vertex $v$ of $G$ the label of the strong component of $v$ in the labeling of $G^{\prime}$. Within an acyclic digraph $G^{\prime}$, let $u \leqslant v$ if $u$ can reach $v$ in $G^{\prime}$ for all $u, v$ in $G^{\prime}$. Clearly ( $G^{\prime}, \leqslant$ ) forms a partially ordered set. Thus, if we have an $f(n)$-bit comparability labeling scheme for posets with planar cover graphs, then we immediately get an $f(n)$-bit reachability scheme for general planar digraphs.

Note that if $\mathcal{C}$ is a class of posets with bounded Boolean dimension, then $\mathcal{C}$ admits an $\mathcal{O}(\log n)$ comparability labeling scheme. To see this, suppose that $\operatorname{bdim}(P) \leqslant d$ for every poset $P$ in $\mathcal{C}$. Now let $P$ be in $\mathcal{C}$, and let $\left(L_{1}, \ldots, L_{d}\right)$ with a formula $\phi$ be a Boolean realizer of $P$. Let $n$ be the number of elements of $P$. We label each element $x \in P$ with a bitstring of length $d \cdot\lceil\log n\rceil$ describing the positions of $x$ in $\left(L_{1}, \ldots, L_{d}\right)$. Now given labels for two elements $x, y \in P$ and the formula $\phi$, we can determine if $x \leqslant y$ in $P$. The formula $\phi$ is a function from $\left(\{0,1\}^{d}\right)^{2}$ to $\{0,1\}$, so there are only $2^{2^{2 d}}$ possibilities, and these can be encoded with an additional $2^{2 d}$ bits in each label. We note that the proof of Theorem 1 gives us conveniently the same $\phi$ for every poset $P$ with a planar cover graph and a unique minimal element, so these extra bits are not necessary.

Therefore, we have the following corollary to Theorem 1, which we believe represents an important step towards a positive resolution of Problem C.

Corollary 3. Planar digraphs with a single source admit an $\mathcal{O}(\log n)$-bit reachability labeling scheme.

## 3. Notation, Terminology and Essential Background Material

We write $[k]$ as a compact form of $\{1, \ldots, k\}$.
When $u$ and $v$ are distinct vertices in a tree $T$, we denote by $u T v$ the unique path in $T$ from $u$ to $v$. This notation allows for the natural notion of concatenation when working with paths, i.e., when $N$ and $N^{\prime}$ are paths in a graph $G, a, b \in N$ and $b, c \in N^{\prime}$, then $a N b N^{\prime} c$ denotes a walk in $G$ formed by concatenating $a N b$ with $b N^{\prime} c$. When this convention is applied later in this paper, the paths $a N b$ and $b N^{\prime} c$ will have no vertices in common other than $b$. As a result, $a N b N^{\prime} c$ will also be a path.

Let $P$ be a poset. When $a$ and $b$ are incomparable in $P$, we will write $a \| b$ in $P$, and sometimes we will just use the short form $a \|_{P} b$. Analogous short forms will be used for $<,>, \leqslant, \geqslant$. When $x \in P$, we let $U_{P}(x)$ consist of all $v \in P$ such that $x<_{P} v$. Also, we set $U_{P}[x]=U_{P}(x) \cup\{x\}$. Of course, the statement that $x_{0}$ is the unique mininal element of $P$ means the same as saying $U_{P}\left[x_{0}\right]$ contains all elements of $P$. Analogously, $D_{P}(x)$ consists of all $u \in P$ such that $u<_{P} x$, and $D_{P}[x]=D_{P}(x) \cup\{x\}$.

When $x y$ is an edge in the cover graph of $P$ and $x<_{P} y$, we say that $x$ is covered by $y$ in $P$, or $y$ covers $x$ in $P$. This constitutes a natural acyclic orientation of the cover graph of $P$, which will be used implicitly.

When $P$ is a poset, $x \in P$, and $S$ is a subset of $P$ that does not contain $x$, we will write $x \|_{P} S$ when $x \|_{P} y$ for every $y \in S$. The notations $x<_{P} S$ and $x>_{P} S$ are defined analogously.

A witnessing path $W$ in $P$ is a sequence $W=\left(u_{0}, \ldots, u_{m}\right)$ of elements in $P$ such that if $m>0$, then $u_{i}$ is covered by $u_{i+1}$ in $P$ whenever $0 \leqslant i<m$. We will say in this case that $W$ is a witnessing path from $u_{0}$ to $u_{m}$. In discussions on sequences of elements, we will use the terms prefix and suffix, i.e., when $\left(z_{0}, \ldots, z_{m}\right)$ is a sequence, and $0 \leqslant i \leqslant m$, the subsequence $\left(z_{0}, \ldots, z_{i}\right)$ is a prefix and the subsequence $\left(z_{i}, \ldots, z_{m}\right)$ is a suffix of the initial sequence.

Members of the ground set of a poset will be called elements and points interchangeably. Also, when we are discussing a graph on the same ground set as a poset, points and elements will also be called vertices.

Let $P$ be a poset, and let $\operatorname{Inc}(P)$ denote the set of all pairs $(a, b)$ such that $a \|_{P} b$. A subset $S$ of $\operatorname{Inc}(P)$ is reversible when there is a linear extension $L$ of $P$ such that $b<a$ in $L$, for all $(a, b) \in S$. Now the dimension of $P$ can be redefined as follows: if $\operatorname{Inc}(P)$ is empty then $\operatorname{dim}(P)=1$ and otherwise $\operatorname{dim}(P)$ is the least positive integer $t$ for which $\operatorname{Inc}(P)$ can be covered by $t$ reversible sets.

A sequence $\left(\left(x_{1}, y_{1}\right), \ldots,\left(x_{k}, y_{k}\right)\right)$ of pairs from $\operatorname{Inc}(P)$ with $k \geqslant 2$ is an alternating cycle of size $k$ if $x_{i} \leqslant P y_{i+1}$ for all $i \in\{1, \ldots, k\}$, cyclically (so $x_{k} \leqslant P y_{1}$ is required). Observe that if $\left(\left(x_{1}, y_{1}\right), \ldots,\left(x_{k}, y_{k}\right)\right)$ is an alternating cycle in $P$, then any subset $I \subseteq \operatorname{Inc}(P)$ containing all the pairs on this cycle is not reversible.

An alternating cycle $\left(\left(x_{1}, y_{1}\right), \ldots,\left(x_{k}, y_{k}\right)\right)$ is strict if we have $x_{i} \leqslant_{P} y_{j}$ if and only if $j=i+1$ (cyclically). Note that in this case, $\left\{x_{1}, \ldots, x_{k}\right\}$ and $\left\{y_{1}, \ldots, y_{k}\right\}$ are $k$-element antichains. Note also that in alternating cycles, we allow that $x_{i}=y_{i+1}$ for some or even all values of $i$. In [17], Trotter and Moore made the following elementary observation that has proven over time to be far reaching in nature: A subset $I \subseteq \operatorname{Inc}(P)$ is reversible if and only if $I$ contains no strict alternating cycle.

Our primary focus will be on posets that have planar cover graphs. When $P$ has a planar cover graph $G$, we fix a drawing without edge crossings of $G$ in the plane, we follow the standard convention that also considers an element $x$ of the poset $P$ as a point in the plane. Similarly, an edge or a path in $G$ will be considered as a subgraph of $G$ and a simply connected set of points in the plane. When $N$ is a path in $G$, the vertices and edges of $N$ form a simply connected set of points in the plane. When $D$ is a cycle in $G$, the points in the plane belonging to the edges of $D$ form a simple closed curve. In this case, we abuse notation by simply saying that a point in the plane is either on $D$, in the interior of $D$ or in the exterior of $D$, without futher reference to the curve.

We start with a compact summary of material which is largely taken from the paper [10] by Kozik, Micek and Trotter. Let $P$ be a poset with a planar cover graph $G$ and with a unique minimal element $x_{0}$.

Since $x_{0}$ is an element of $P$, we can fix a drawing without edge crossings of $G$ with $x_{0}$ on the exterior face. To assist in ordering edges in arguments to follow, we append an imaginary edge $e_{-\infty}$ attached to $x_{0}$ in the exterior face. This setup is illustrated in Figure 2, and we will return to this example several times later in the paper.


Figure 2. A poset with a planar cover and a unique minimal element $x_{0}$ drawn in the exterior face. An imaginary edge $e_{-\infty}$ is attached to $x_{0}$. In this figure, the black and red edges are oriented left-to-right in the plane, while the blue and green edges are oriented right-to-left.

When $z$ is an element of $P$, there is a natural clockwise cyclic ordering of the edges of $G$ incident with $z$. When $e_{0}, e$, and $e^{\prime}$ are edges (not necessarily distinct) incident to $z$, we will write $e_{0} \preccurlyeq e \preccurlyeq e^{\prime}$, if starting with edge $e_{0}$ and proceeding in a clockwise manner around $z$, stopping at $e^{\prime}$, we have visited the edge $e$. We can replace one or both of the $\preccurlyeq$ symbols with $\prec$ when the corresponding edges are distinct.

When $e_{0}$ is a particular edge incident to $z$, there is a clockwise linear order on the edges incident with $z$ with $e_{0}$ the least element. If $e$ and $e^{\prime}$ are distinct edges and neither is $e_{0}$, we say that $e \prec e^{\prime}$ in the $\left(z, e_{0}\right)$-ordering if $e_{0} \prec e \prec e^{\prime}$. To reinforce the geometric implications, we will also say that $e$ is left of $e^{\prime}$ in the $\left(z, e_{0}\right)$-ordering. Of course, we will also say that $e^{\prime}$ is right of $e$ in the $\left(z, e_{0}\right)$-ordering when $e$ is left of $e^{\prime}$ in the $\left(z, e_{0}\right)$-ordering. The $\left(z, e_{0}\right)$-ordering on edges is illustrated in Figure 3.

Let $u$ and $u^{\prime}$ be (not necessarily distinct) elements of $P$. Also, let $W$ and $W^{\prime}$ be paths (not necessarily witnessing paths) in $P$ from $x_{0}$ to $u$ and $u^{\prime}$, respectively. We say that $W$ and $W^{\prime}$ are $x_{0}$-consistent if there is an element $z$ common to $W$ and $W^{\prime}$ such that (1) $x_{0} W z=x_{0} W^{\prime} z$; and (2) $z W u$ and $z W^{\prime} u^{\prime}$ are disjoint except their common starting element $z$. Note that $W$ and $W^{\prime}$ are $x_{0}$-consistent whenever one is a prefix of the other. In fact, a path $W$ from $x_{0}$ to an element $u$ in $P$ is $x_{0}$-consistent with itself.

Now suppose that $W$ and $W^{\prime}$ are $x_{0}$-consistent and neither is a prefix of the other. Let $z$ be the largest element of $P$ common to both $W$ and $W^{\prime}$, and let $e_{0}$ be the edge immediately before $z$ on the path $x_{0} W z$ (let $e_{0}=e_{-\infty}$ if $z=x_{0}$ ). Let $e$ and $e^{\prime}$ be the first edges of $z W u$ and $z W^{\prime} u^{\prime}$, respectively. We say that $W$ is $x_{0}$-left of $W^{\prime}$ if $e$ is left of $e^{\prime}$ in the $\left(z, e_{0}\right)$-ordering. Symmetrically, we say that $W$ is $x_{0}$-right of $W^{\prime}$ if $e$ is right of $e^{\prime}$ in the $\left(z, e_{0}\right)$-ordering. Note that if $W$ and $W^{\prime}$ are $x_{0}$-consistent, and neither is a prefix of the other, then either $W$ is $x_{0}$-left of $W^{\prime}$, or $W$ is $x_{0}$-right of $W^{\prime}$.


Figure 3. Left: For the vertex $z$ and the fixed edge $e_{0}$ incident with $z$, edge $e$ is left of $e^{\prime}$ in the $\left(z, e_{0}\right)$-ordering. Right: The paths $W$ and $W^{\prime}$ are $x_{0}$-consistent and $W$ is $x_{0}$-left of $W^{\prime}$.

## 4. The Common Part of the Proofs of our Two Main Theorems

Let $P$ be a poset with a planar cover graph and a unique minimal element. We will quickly develop a classification scheme that results in every incomparable pair in $P$ being labeled as one of four types: left, right, outside, or inside. We will then show that there are linear extensions $L_{1}$ and $L_{2}$ of $P$ such that if $(a, b) \in \operatorname{Inc}(P)$, and $(a, b)$ is a left pair, a right pair or an outside pair, then $b<a$ in either $L_{1}$ or $L_{2}$.

Now suppose our focus was on Boolean dimension. Given a pair $(a, b)$ of distinct points in $P$, our goal is to answer with full certainty the question: "Is $a$ less or equal than $b$ in $P$ ?". Suppose that we know the answer as to whether $a \leqslant b$ in $L_{i}$ for each $i \in[2]$. If any one of these answers is negative, then we know that $a$ is not less or equal than $b$ in $P$. Moreover, if the answer is positive for each $i \in[2]$, then we know that one of the following two statements hold: (1) $a$ is less than $b$ in $P$; or $(2)(a, b)$ is an incomparable pair that is an inside pair. Our goal for part 2 of the proof for Theorem 1, given in Section 5, will be to show that we can construct 11 additional linear orders $\left(L_{3}, \ldots, L_{13}\right)$, so that if we also know the answer to whether $a$ is less
or equal than $b$ in $L_{j}$ for $3, \ldots, 13$, then we can answer with complete confidence the question as to whether $a$ is less or equal than $b$ in $P$.

Now suppose that our focus was on the issue of dim-boundedness for posets with planar cover graphs and a unique minimal element. In the second part of this proof, given in Section 6, we will then show that the set of all inside pairs can be covered with $2 \mathrm{se}(P)$ reversible sets.

Now we begin our work in the common part of the proof by developing a series of propositions and lemmas concerning posets that have a unique minimal element and a planar cover graph. Accordingly, we continue with the same setup used in the preceding section (and as illustrated in Figure 2, i.e., we assume:
(i) $P$ is a poset and $x_{0}$ is the unique minimal element of $P$.
(ii) $G$ is the cover graph of $P$ and $G$ is planar.
(iii) We fix a drawing without edge crossings of $G$ in the plane with $x_{0}$ on the exterior face.
(iv) To provide a base edge among those incident in $G$ with $x_{0}$, we add to the drawing an "imaginary" edge $e_{-\infty}$ in the exterior face, with $e_{-\infty}$ incident with $x_{0}$.

Let $u$ be an element of $P$. The leftmost witnessing path from $x_{0}$ to $u$, denoted by $W_{L}(u)$, is constructed using the following inductive procedure: If $u=x_{0}$ then $W_{L}(u)$ is a one-vertex path containing $u$. Otherwise, start with $u_{0}=x_{0}$ and $e_{0}=e_{-\infty}$. Then $u_{0}<_{P} u$. Now suppose that for some $i \geqslant 0$, we have defined a witnessing path $W_{i}=\left(u_{0}, \ldots, u_{i}\right), u_{i}<u$ in $P$, and $e_{i}$ is the edge immediately before $u_{i}$ on $W_{i}$. Among the edges incident with $u_{i}$, some may be directed towards $u_{i}$, but since $u_{i}<_{P} u$, there is a non-empty set $E_{i}$ of edges incident with $u_{i}$ that are directed away from $u_{i}$ and lie on a witnessing path from $u_{i}$ to $u$. We let $e_{i+1}$ be the leftmost edge in $E_{i}$ in the $\left(u_{i}, e_{i}\right)$-ordering, and we take $u_{i+1}$ as the other end point of $e_{i+1}$. When $u_{i+1}=u$, the procedure halts and outputs the witnessing path $\left(u_{0}, \ldots, u_{i+1}\right)$ as $W_{L}(u)$. For the poset shown in Figure 2, we illustrate the leftmost path from $x_{0}$ to $z$ in Figure 4.


Figure 4. We show $W_{L}(z)$, the leftmost witnessing path from $x_{0}$ to $z$, using bold edges. The remaining edges in the cover graph are dashed. In this figure, the black and red edges are oriented left-to-right in the plane, while the blue and green edges are oriented right-to-left.

Proposition 4. Let $u$ and $u^{\prime}$ be elements of $P$. Then $W_{L}(u)$ and $W_{L}\left(u^{\prime}\right)$ are $x_{0}$-consistent. Also, $W_{R}(u)$ and $W_{R}\left(u^{\prime}\right)$ are $x_{0}$-consistent.

Proof. We use an argument by contradiction to show that $W_{L}(u)$ and $W_{L}\left(u^{\prime}\right)$ are $x_{0}$-consistent. The argument for $W_{R}(u)$ and $W_{R}\left(u^{\prime}\right)$ is symmetric. Let $z$ be the largest element of $P$ such that $x_{0} W_{L}(u) z=x_{0} W_{L}\left(u^{\prime}\right) z$. Since $W_{L}(u)$ and $W_{L}\left(u^{\prime}\right)$ are not $x_{0}$-consistent, (1) $z \notin\left\{u, u^{\prime}\right\}$; (2) there are distinct elements $w$ and $w^{\prime}$ both of which cover $z$ in $P$ such that $W_{L}(u)$ contains the edge $e=z w$ and $W_{L}\left(u^{\prime}\right)$ contains the edge $e^{\prime}=z w^{\prime}$; and (3) there is an element $z^{\prime} \in P$ common to $W_{L}(u)$ and $W_{L}\left(u^{\prime}\right)$ such that $\left\{w, w^{\prime}\right\}<_{P} z^{\prime}$. Note that $z^{\prime} \leqslant P\left\{u, u^{\prime}\right\}$.

Let $e_{0}$ be the last edge common to $x_{0} W_{L}(u) z=x_{0} W_{L}\left(u^{\prime}\right) z$ (again $e_{0}=e_{-\infty}$ if $z=x_{0}$ ). Without lost of generality, we may suppose that $e$ is left of $e^{\prime}$ in the $\left(z, e_{0}\right)$-ordering. Now consider the choice the inductive procedure made in constructing $W_{L}\left(u^{\prime}\right)$ made when it was at vertex $z$. It considered all edges of the form $z v$ with $v$ covering $z$ in $P$ and $v \leqslant P u^{\prime}$. Among these edges, the algorithm chooses the edge which is least in the $\left(z, e_{0}\right)$-ordering. However, the edge $e=z w$ is among the choices available since $w<z^{\prime} \leqslant u^{\prime}$ in $P$. It follows that the procedure will not choose the edge $e^{\prime}$. The contradiction completes the proof.

Proposition 5. Let $u$ and $v$ be elements of $P$. Then the following statements hold:
(i) If $W_{L}(u)$ is $x_{0}$-left of $W_{L}(v)$, then $u \nless v$ in $P$.
(ii) If $W_{R}(u)$ is $x_{0}$-right of $W_{R}(v)$, then $u \nless v$ in $P$.

Proof. We prove statement (i). The argument for statement (ii) is symmetric. Let $z$ be the largest element of $P$ common to $W_{L}(u)$ and $W_{L}(v)$. Also, let $e_{0}$ be the common edge of these two paths that is immediately before $z$ (let $e_{0}=e_{-\infty}$ if $z=x_{0}$ ). Let $e=z w$ and $e^{\prime}=z w^{\prime}$ be the edges immediately after $z$ in $W_{L}(u)$ and $W_{L}(v)$, respectively. Since $W_{L}(u)$ is $x_{0}$-left of $W_{L}(v)$, we know that $e$ is left of $e^{\prime}$ in the ( $z, e_{0}$ )-ordering.

Now suppose that the proposition fails with $u<v$ in $P$. This implies $w \leqslant u<v$ in $P$. Therefore, in leaving $z$, the procedure for constructing $W_{L}(v)$ had available the edge $e$, but incorrectly chose the edge $e^{\prime}$. The contradiction completes the proof.

There are two versions of the following proposition, with the roles of left and right interchanged. We state and prove one of the two versions.

Proposition 6. Let $v$ and $z$ be elements of $P$ such that $z$ is on $W_{L}(v)$ with $z \neq v$. Let $e^{+}$ and $e^{-}$be the edges of $W_{L}(v)$ that are, respectively, immediately after and immediately before $z$ (if $z=x_{0}$ then $e^{-}=e_{-\infty}$ ). If $z$ is covered by $u$ in $P$, and the edge $e=z u$ is left of $e^{+}$in the $\left(z, e^{-}\right)$-ordering, then $W_{L}\left(u^{\prime}\right)$ is $x_{0}$-left of $W_{L}(v)$ for all $u^{\prime} \in U_{P}[u]$.

Proof. Consider the common prefix of $W_{L}\left(u^{\prime}\right)$ and $W_{L}(v)$. We split into two cases: either the last element $w$ of their common prefix satisfies $w<z$ in $P$, or $w \geqslant z$ in $P$.

Assume first that $w<z$ in $P$. Let $e_{w}^{+}$and $e_{w}^{-}$be the edges of $W_{L}(v)$ that are, respectively, immediately after and immediately before $w$ (if $z=x_{0}$ then $e_{w}^{-}=e_{-\infty}$ ). Consider the construction process of $W_{L}\left(u^{\prime}\right)$ at element $w$. It looks for the least edge $e^{\prime}=w w^{\prime}$ in the $\left(w, e_{w}^{-}\right)$-ordering such that $w<w^{\prime} \leqslant u^{\prime}$ in $P$. Since $e_{w}^{+}$is a valid choice and the two paths $W_{L}\left(u^{\prime}\right)$ and $W_{L}(v)$ split at $w$, the path $W_{L}\left(u^{\prime}\right)$ must have found a better candidate. Therefore, $W_{L}\left(u^{\prime}\right)$ is $x_{0}$-left of $W_{L}(v)$, in this case.

Now assume that the common prefix of $W_{L}\left(u^{\prime}\right)$ and $W_{L}(v)$ contains $z$, i.e., $w \geqslant z$ in $P$. Again, consider the construction process of $W_{L}\left(u^{\prime}\right)$, this time at element $z$. Since $e$ is left of $e^{+}$in the
$\left(z, e^{-}\right)$-ordering and $z<u \leqslant u^{\prime}$ in $P$, we conclude that there are better choices than $e^{+}$for $W_{L}\left(u^{\prime}\right)$ to continue from $z$. Therefore, the two paths $W_{L}\left(u^{\prime}\right)$ and $W_{L}(v)$ split at $z$ and $W_{L}\left(u^{\prime}\right)$ is $x_{0}$-left of $W_{L}(v)$, as desired.

It is easy to construct examples of witnessing paths $W, W^{\prime}, W^{\prime \prime}$ from $x_{0}$ to $u, u^{\prime}, u^{\prime \prime}$ such that (1) $W$ and $W^{\prime}$ are $x_{0}$-consistent and $W$ is $x_{0}$-left of $W^{\prime} ;(2) W^{\prime}$ and $W^{\prime \prime}$ are $x_{0}$-consistent and $W^{\prime}$ is $x_{0}$-left of $W^{\prime \prime}$; and (3) $W$ and $W^{\prime \prime}$ are not $x_{0}$-consistent. However, if we know that $W$ and $W^{\prime \prime}$ are $x_{0}$-consistent, we have the following more palatable result.

Proposition 7. Let $W, W^{\prime}, W^{\prime \prime}$ be witnessing paths from $x_{0}$ to $u, u^{\prime}, u^{\prime \prime}$, respectively, and suppose that any two of these paths are $x_{0}$-consistent. Then the following statements hold:
(i) If $W$ is $x_{0}$-left of $W^{\prime}$, and $W^{\prime}$ is $x_{0}$-left of $W^{\prime \prime}$, then $W$ is $x_{0}$-left of $W^{\prime \prime}$.
(ii) If $W$ is $x_{0}$-right of $W^{\prime}$, and $W^{\prime}$ is $x_{0}$-right of $W^{\prime \prime}$, then $W$ is $x_{0}$-right of $W^{\prime \prime}$.

Proof. We give the argument for statement (i). The argument for statement (ii) is symmetric. Let $z$ be the largest element common to all three paths $W, W^{\prime}, W^{\prime \prime}$ (in case $z=x_{0}$, let $\left.e_{0}=e_{-\infty}\right)$. Let $e_{0}$ be the edge immediately before $z$ on all three paths. Then let $e, e^{\prime}, e^{\prime \prime}$ be the edge after $e_{0}$ on $W, W^{\prime}, W^{\prime \prime}$, respectively. The cyclic ordering on these edges is:

$$
e_{0} \prec e \preccurlyeq e^{\prime} \preccurlyeq e^{\prime \prime} \prec e_{0}
$$

Furthermore, either $e \neq e^{\prime}$ or $e^{\prime} \neq e^{\prime \prime}$. It follows that

$$
e_{0} \prec e \prec e^{\prime \prime} \prec e_{0}
$$

This together with the fact that $W$ and $W^{\prime \prime}$ are $x_{0}$-consistent implies that $W$ is $x_{0}$-left of $W^{\prime \prime}$, as desired. With this observation, the proof is complete.
4.1. Four Types of Incomparable Pairs. Let $u$ and $v$ be elements of $P$.

- We say $(u, v)$ is a left pair in $P$ if
$W_{L}(u)$ is $x_{0}$-left of $W_{L}(v)$ and $W_{R}(u)$ is $x_{0}$-left of $W_{R}(v)$.
- We say $(u, v)$ is a right pair in $P$ if
$W_{L}(u)$ is $x_{0}$-right of $W_{L}(v)$ and $W_{R}(u)$ is $x_{0}$-right of $W_{R}(v)$;
Proposition 8. Let $u, v, w$ be elements of $P$. Then the following statements hold:
(i) If $u<v$ in $P$, then $(u, v)$ is not a left pair.
(ii) If $u<v$ in $P$, then $(u, v)$ is not a right pair.
(iii) If $u$ and $v$ are comparable, then $(u, v)$ is not a left pair, and $(u, v)$ is not a right pair.
(iv) If $(u, v)$ is a left pair, and $(v, w)$ is a left pair, then $(u, w)$ is a left pair.
(v) If $(u, v)$ is a right pair, and $(v, w)$ is a right pair, then $(u, w)$ is a right pair.

Proof. Statements (i) and (ii) follow immediately from the two statements of Proposition 5. Statement (iii) is then an immediate consequence of statements (i) and (ii).

By Proposition 4 we have that any two paths from $\left\{W_{L}(u), W_{L}(v), W_{L}(w)\right\}$ are $x_{0}$-consistent and any two paths from $\left\{W_{R}(u), W_{R}(v), W_{R}(w)\right\}$ are $x_{0}$-consistent. Statements (iv) and (v) then follow by applying Proposition 7 .

Among the incomparable pairs of $P$, we characterize those that are neither left pairs nor right pairs as follows. Let $(u, v) \in \operatorname{Inc}(P)$.

- We say $(u, v)$ is an outside pair in $P$ if
$W_{L}(u)$ is $x_{0}$-left of $W_{L}(v)$ and $W_{R}(u)$ is $x_{0}$-right of $W_{R}(v)$.
- We say $(u, v)$ is an inside pair in $P$ if
$W_{L}(u)$ is $x_{0}$-right of $W_{L}(v)$ and $W_{R}(u)$ is $x_{0}$-left of $W_{R}(v)$;
Evidently, $(u, v)$ is an outside pair if and only if $(v, u)$ is an inside pair. Also, when $(u, v) \in$ $\operatorname{Inc}(P)$, then the pair $(u, v)$ is one of the four mutually exclusive types: left, right outside, inside. However, in contrast to the situation with left and right pairs, it may happen that a comparable pair satisfies the requirements to be an inside pair.

Example 9. For the poset whose cover graph is shown in Figure 2:
(i) Among the left pairs are: $(a, e),(h, m),(z, x)$ and $(p, t)$.
(ii) Among the right pairs are: $(c, a),(d, b),(k, h)$, and $(t, q)$.
(iii) Among the outside pairs are: $(m, b),(s, v)$ and $(z, q)$.
(iv) Among the inside pairs are: $(c, g),(b, x),(m, p)$ and $(q, z)$.

We introduce the following notation:

- $X$ (left) denotes the set of all left pairs of $P$.
- $X$ (right) denotes the set of all right pairs of $P$.
- $X$ (outside) denotes the set of all incomparable pairs of $P$ that are outside pairs.

Observe that all three sets $X$ (left), $X$ (right), $X$ (outside) are sets of incomparable pairs in $P$.
Proposition 10. The following two subsets of $\operatorname{Inc}(P)$ are reversible:
(i) $X_{1}=X$ (left) $\cup X$ (outside).
(ii) $X_{2}=X$ (right) $\cup X$ (outside).

Proof. We prove that $X_{1}$ is reversible. The argument for $X_{2}$ is symmetric. Arguing by contradiction, we assume that $k \geqslant 2$ and $\left(\left(a_{1}, b_{1}\right), \ldots,\left(a_{k}, b_{k}\right)\right)$ is a strict alternating cycle of pairs from $X_{1}$. Now let $\alpha$ be any integer in $[k]$. Since $\left(a_{\alpha}, b_{\alpha}\right) \in X_{1}$, we know that $W_{L}\left(a_{\alpha}\right)$ is $x_{0}$-left of $W_{L}\left(b_{\alpha}\right)$. Since $a_{\alpha} \leqslant b_{\alpha+1}$ in $P$, Proposition 5 implies that either $W_{L}\left(a_{\alpha}\right)$ is a prefix of $W_{L}\left(b_{\alpha+1}\right)$, or $W_{L}\left(a_{\alpha}\right)$ is $x_{0}$-right of $W_{L}\left(b_{\alpha+1}\right)$. Then Proposition 7 implies that $W_{L}\left(b_{\alpha+1}\right)$ is $x_{0}$-left of $W_{L}\left(b_{\alpha}\right)$. Clearly, this statement cannot hold for all $\alpha \in[k]$, cyclically. The contradiction completes the proof.

Readers may note that the preceding proposition asserts the existence of two linear extensions reversing specified sets of incomparable pairs. In fact, the two linear extensions $L_{1}, L_{2}$ are uniquely determined:
(i) Set $a<b$ in $L_{1}$ if $W_{L}(a)$ is a prefix of $W_{L}(b)$, or $W_{L}(b)$ is $x_{0}$-left of $W_{L}(a)$.
(ii) Set $a<b$ in $L_{2}$ if $W_{R}(a)$ is a prefix of $W_{R}(b)$, or $W_{R}(b)$ is $x_{0}$-right of $W_{R}(a)$.

Then $L_{1}$ is a linear extension reversing all pairs in $X$ (left) $\cup X$ (outside). Also, $L_{2}$ is a linear extension reversing all pairs in $X$ (right) $\cup X$ (outside). We have elected to give the proof
in terms of alternating cycles since this approach will be our primary tool for working with reversible sets in the remainder of the paper. Independent of the proof, Proposition 10 implies that it is essential that we understand more fully the distinctive characteristics of inside pairs.
4.2. Shadows, Shadow Depth and Shadow Sequences. Let $W$ and $W^{\prime}$ be a pair of witnessing paths. We will say that a set $\mathcal{B}$ of points in the plane is the non-degenerate block enclosed by $W$ and $W^{\prime}$ if the following three conditions are met: (1) there are distinct elements $x$ and $y$ of $P$ such that $x<_{P} y, x$ is not covered by $y$, and both $W$ and $W^{\prime}$ are witnessing paths from $x$ to $y$; (2) $W \cap W^{\prime}=\{x, y\}$ so $W \cup W^{\prime}$ is a cycle $D$ in the cover graph $G$; and (3) $\mathcal{B}$ is the set of points in the plane that are on or inside $D$.

On the other hand, we will say that a set $\mathcal{B}$ of points in the plane is the degenerate block enclosed by $W$ and $W^{\prime}$ if the following three conditions are met: (1) there are elements $x$ and $y$ of $P$ such that $x$ is covered by $y$ in $P ;(2) W=W^{\prime}$ is the 2-element witnessing path from $x$ to $y$ consisting of a single edge; and (3) $\mathcal{B}$ is the set of points in the plane on the edge in $G$ between $x$ and $y$.

We note that in both cases, the set $\mathcal{B}$ and the elements $x$ and $y$ are determined by the pair ( $W, W^{\prime}$ ), so we will simply say that $\mathcal{B}$ is the block enclosed by $W$ and $W^{\prime}$, thereby allowing $\mathcal{B}$ to be either degenerate or non-degenerate. Also, we will refer to the element $x$ as $\min (\mathcal{B})$, while the element $y$ will be $\max (\mathcal{B})$. When $\mathcal{B}$ is the block enclosed by $W$ and $W^{\prime}$, it is important to note that if $w$ is an element of $P$ on the boundary of $\mathcal{B}$, i.e., $w \in W \cup W^{\prime}$, then $w \leqslant \max (\mathcal{B})$ in $P$. Soon it will become clear that if $u$ is any element of $P$ that is the interior of $\mathcal{B}$, then $\min (\mathcal{B})<u$ in $P$. However, in general, we will not know the order relation, if any, between $u$ and $\max (\mathcal{B})$.

Let $\mathcal{B}$ be a non-degenerate block enclosed by a pair $\left(W, W^{\prime}\right)$ of witnessing paths from $\min (\mathcal{B})$ to $\max (\mathcal{B})$. We will designate one of $W$ and $W^{\prime}$ to be the left side of $\mathcal{B}$, while the other path will be the right side of $\mathcal{B}$. The distinction is made using the following convention. Consider a clockwise traversal of a portion of the boundary of $\mathcal{B}$, starting at $\min (\mathcal{B})$ and stopping at $\max (\mathcal{B})$. Then the path we have followed is the left side of $\mathcal{B}$; the other side is the right side of $\mathcal{B}$. The elements $\min (\mathcal{B})$ and $\max (\mathcal{B})$ belong to both sides. An element $z$ on the left side of $\mathcal{B}$, with $\min (\mathcal{B})<z<\max (\mathcal{B})$ in $P$ is said to be strictly on the left side of $\mathcal{B}$. An element $z$ on the right side of $\mathcal{B}$, with $\min (\mathcal{B})<z<\max (\mathcal{B})$ in $P$ is said to be strictly on the right side of $\mathcal{B}$.

When $\mathcal{B}$ is a degenerate block consisting of a single edge, we consider this edge to be both the left side and the right side of $\mathcal{B}$, and there are no elements that are strictly on one of the two sides.

For the discussion to follow, we fix an element $z \in U_{P}\left(x_{0}\right)$. Then we will introduce notation that defines sequences, paths, blocks, and elements of $P$, all depending on the choice of $z$. However, to maintain some reasonable level in the complexity of notation, we will not indicate the dependence on $z$. With $z$ fixed, we let $W_{L}=W_{L}(z), W_{R}=W_{R}(z)$. Then $W_{L} \cap W_{R}$ is a chain containing $x_{0}$ and $z$. Accordingly, the elements of $W_{L} \cap W_{R}$ can be listed sequentially as $\left(z_{0}, \ldots, z_{m}\right)$ such that $i<j$ if $z_{i}<_{P} z_{j}$. We will refer to $\left(z_{0}, \ldots, z_{m}\right)$ as the sequence of common points of $z$. Note that $m \geqslant 1, z_{0}=x_{0}$, and $z_{m}=z$. For all $i \in[m]$, we define $\mathcal{B}_{i}$ to be the block enclosed by $z_{i-1} W_{L} z_{i}$ and $z_{i-1} W_{R} z_{i}$. We also call $\mathcal{B}_{i}$ the block between $z_{i-1}$ and $z_{i}$. Each block in the sequence $\left(\mathcal{B}_{1}, \ldots, \mathcal{B}_{m}\right)$ is either degenerate or non-degenerate. In general, any of the possible $2^{m}$ outcomes from this binary distinction is possible.

We now make some elementary but nonetheless important observations about the sequence $\left(\mathcal{B}_{1}, \ldots, \mathcal{B}_{m}\right)$. Suppose that $m \geqslant 2$, and let $i$ be an integer with $0<i<m$. Since $i<m$, we know $z_{i} \neq z$, and therefore $z_{i}<_{p} z$. Since $w \leqslant{ }_{P} z_{i}$ for all elements $w$ of $P$ that are on the boundary of $\mathcal{B}_{i}$, we know $z$ is not on the boundary of $\mathcal{B}_{i}$. We then have two cases:

Case 1. $z$ is in the exterior of $\mathcal{B}_{i}$.
In this case, it follows that if $W$ is a witnessing path from $z_{i}$ to $z$, all edges and vertices of $W$, except the starting point $z_{i}$ are in the exterior of $\mathcal{B}_{i}$. We apply this reasoning twice, first when $W=z_{i} W_{L} z$, and second when $W=z_{i} W_{R} z$. In particular, we conclude that the boundaries of $\mathcal{B}_{i}$ and $\mathcal{B}_{i+1}$ stay disjoint apart from $z_{i}$. Note also that $\mathcal{B}_{i+1}$ cannot contain $\mathcal{B}_{i}$ as in this case every witnessing path from $x_{0}$ to $z_{i}$ would intersect the boundary of $\mathcal{B}_{i+1}$ but all the elements at the boundary of $\mathcal{B}_{i+1}$ are larger than $z_{i}$ in $P$. We conclude that (1) $z_{i}$ is the only point in the plane common to $\mathcal{B}_{i}$ and $\mathcal{B}_{i+1}$; and (2) if $i+1<j \leqslant m$, then $\mathcal{B}_{i}$ and $\mathcal{B}_{j}$ are disjoint.

Case 2. $z$ is in the interior of $\mathcal{B}_{i}$.
In this case, it follows that if $W$ is a witnessing path from $z_{i}$ to $z$, all edges and vertices of $W$, except the starting point $z_{i}$ are in the interior of $\mathcal{B}_{i}$. Again, we apply this reasoning twice, first when $W=z_{i} W_{L} z$, and second when $W=z_{i} W_{R} z$. Now we conclude that (1) $z_{i}$ is the only point in the plane common to the boundaries of $\mathcal{B}_{i}$ and $\mathcal{B}_{i+1} ;(2)$ all points in the plane belonging to $\mathcal{B}_{i+1}$, except $z_{i}$, are in the interior of $\mathcal{B}_{i}$; and (3) if $i+1<j \leqslant m$, then all points of $\mathcal{B}_{j}$ are in the interior of $\mathcal{B}_{i}$.

When this second case holds, and $z$ is in the interior of $\mathcal{B}_{i}$, we will call the element $z_{i}$ a reversing element for $z$. The number of reversing elements in the sequence of common elements of $z$ will be called the shadow depth of $z$, denoted $\operatorname{sd}(z)$.

Next, when $\operatorname{sd}(z)=r$, we define a sequence $\left(\operatorname{shad}_{0}(z), \ldots, \operatorname{shad}_{r}(z)\right)$ of sets called the shadow sequence of $z$. First, let $\left(i_{1}, \ldots, i_{r}\right)$ be the subsequence of $(1, \ldots, m-1)$ determined by the subscripts of the reversing elements of $z$. We expand this sequence by adding $i_{0}=0$ at the beginning, and then adding $i_{r+1}=m$ at the end. For each $j$ with $0 \leqslant j \leqslant r$, we then set

$$
\operatorname{shad}_{j}(z)=\bigcup_{i_{j}<i \leqslant i_{j+1}} \mathcal{B}_{i}
$$

Also, we refer to $\left(\mathcal{B}_{i_{j+1}}, \ldots, \mathcal{B}_{i_{j+1}}\right)$ as the sequence of blocks of the $j$-shadow of $z$. We call $\mathcal{B}_{i_{j+1}}$ the terminal block of $\operatorname{shad}_{j}(z)$. Also, we call the element call $z_{i_{j}}$ the base element of $\operatorname{shad}_{j}(z)$, and we call $z_{i_{j+1}}$ the terminal element of $\operatorname{shad}_{j}(z)$. These definitions are illustrated in Figure 5. Note that this is our familiar poset from Figure 2, with certain elements relabeled.

We have already made some basic observations concerning shadows, shadow sequences and shadow depth. Now we give a series of four additional propositions concerning these concepts. The first of these includes only some nearly self-evident statements listed explicitly for emphasis, so no additional arguments are given. However, three more substantive propositions follow.

Proposition 11. Let $z \in U_{P}\left(x_{0}\right)$, let $\left(z_{0}, \ldots, z_{m}\right)$ be the sequence of common points of $z$. If $i \in[m]$, then the sequence of common elements of $z_{i}$ is $\left(z_{0}, \ldots, z_{i}\right)$. Furthermore, if $j$ is an integer with $0 \leqslant j \leqslant \operatorname{sd}(z)$, and $\mathcal{B}$ is a block of $\operatorname{shad}_{j}(z)$. Then the following statements hold:
(i) If $x$ is the base element of $\operatorname{shad}_{j}(z)$, then $\operatorname{sd}(x)=j-1$, unless $j=0$ and $x=x_{0}$.


Figure 5. The sequence of common points of $z=z_{7}$ is $\left(z_{0}, \ldots, z_{7}\right)$. The shadow depth of $z$ is 3 , and its reversing elements are $z_{2}, z_{4}$, and $z_{6}$. Shadow blocks $\mathcal{B}_{3}, \mathcal{B}_{5}, \mathcal{B}_{7}$ are degenerate, while $\mathcal{B}_{1}, \mathcal{B}_{2}, \mathcal{B}_{4}, \mathcal{B}_{6}$ are non-degenerate. In this figure, the black and red edges are oriented left-to-right in the plane, while the blue and green edges are oriented right-to-left.
(ii) $\operatorname{shad}_{j}(\max (\mathcal{B})) \subseteq \operatorname{shad}_{j}(z)$, with equality holding if and only if $\mathcal{B}$ is the terminal block of $\operatorname{shad}_{j}(z)$.

Let $\mathcal{B}$ be a block. We will refer to $\mathcal{B}$ as a shadow block when there is an element $z \in U_{P}\left(x_{0}\right)$ such that if $\left(z_{0}, \ldots, z_{m}\right)$ is the sequence of common points of $z$, then there is some $i \in[m]$ such that $\mathcal{B}$ is the block between $z_{i-1}$ and $z_{i}$.

Proposition 12. Let $\mathcal{B}$ be a shadow block. If $u$ is an element of $P$ that belongs to $\mathcal{B}$, then $\min (\mathcal{B})$ belongs to both $W_{L}(u)$ and $W_{R}(u)$.

Proof. To simplify the notation for the argument, we set $x=\min (\mathcal{B})$, and $y=\max (\mathcal{B})$. Our goal is to show that $x$ belongs to both $W_{L}(u)$ and $W_{R}(u)$. We provide full details to show that $x$ belongs to $W_{L}(u)$. The argument for $W_{R}(u)$ is symmetric.

Since $x_{0}$ is in the exterior face and $u$ is in $\mathcal{B}$, the path $W_{L}(u)$ intersects the boundary of $\mathcal{B}$. Let $w$ be the least element of $P$ that is on $W_{L}(u)$ and is in $\mathcal{B}$. In particular, $W_{L}(w)$ is a prefix of $W_{L}(u)$ and $w$ is on the boundary of $\mathcal{B}$. If $w$ is on the left side of $\mathcal{B}$, i.e. $w \in x W_{L}(y) y$, then $W_{L}(w)$ is a prefix of $W_{L}(y)$. Thus $x \in W_{L}(u)$, as desired.

We may therefore assume that $w$ is strictly on the right side of $\mathcal{B}$ and argue for contradiction. In particular, we have $x<w<y$ in $P$.

Now consider the $x_{0}$-consistent paths $W_{L}(x)$ and $W_{L}(w)$. The comparability $x<w$ in $P$ and Proposition 5 imply that $W_{L}(w)$ is $x_{0}$-left of $W_{L}(x)$. Let $v$ be the largest element of $P$ common to $W_{L}(w)$ and $W_{L}(x)$. Let $e_{0}$ be the edge common to $W_{L}(w)$ and $W_{L}(x)$ that is immediately before $v$, if $v=x_{0}$ then $e_{0}=e_{-\infty}$. Then let $e_{w}=v w^{\prime}$ be the edge on $W_{L}(w)$ that is immediately after $v$, and let $e_{x}=v x^{\prime}$ be the edge of $W_{L}(x)$ that is immediately after $v$. Then $e_{w}$ is left of $e_{x}$ in the $\left(v, e_{0}\right)$-ordering.

Recall that since $\mathcal{B}$ is a shadow block $W_{L}(x)$ is a prefix of $W_{L}(y)$. Therefore, $v<x^{\prime} \leqslant x<y$ in $P$ and all these elements lie on $W_{L}(y)$. However, $v<w<y$ in $P$ as well and therefore the
construction procedure of $W_{L}(y)$ after reaching $v$ should prefer $e_{w}$ over $e_{x}$. This contradicts the assumption that $w$ is strictly on the right side of $\mathcal{B}$.

Proposition 13. Let $\mathcal{B}$ be a shadow block. If $W$ is a witnessing path in $P$, and both the starting and the ending point of $W$ are in $\mathcal{B}$, then all edges of $W$ are in $\mathcal{B}$.

Proof. The proposition holds trivially if $\mathcal{B}$ is degenerate, so we will assume that $\mathcal{B}$ is nondegenerate. Then we argue by contradiction. Among all witnessing paths for which the proposition fails, we take $W$ as one of minimum length. Let $u$ and $v$ be the starting and ending points, respectively, of $W$. Then $u$ and $v$ are on the boundary of $\mathcal{B}$, while all edges of $W$, and any vertices of $W$ that are between $u$ and $v$ are in the exterior of $\mathcal{B}$. The notation for the remainder of the argument is simplified by setting $x=\min (\mathcal{B})$, and $y=\max (\mathcal{B})$. Then $x \leqslant u<v \leqslant y$ in $P$. So $u \neq y$ and $v \neq x$. Let $N$ be the uniquely determined (and possibly trivial) witnessing path from $v$ to $y$ that is a suffix of one of the two sides of $\mathcal{B}$. Let $e$ be the first edge of $W$.

Since $\mathcal{B}$ is a shadow block, the left side of $\mathcal{B}$ is contained in $W_{L}(y)$ and the right side of $\mathcal{B}$ is contained in $W_{R}(y)$. If $u$ is on the left side of $\mathcal{B}$, let $e_{L}^{+}$and $e_{L}^{-}$be, respectively, the edges immediately after and immediately before $u$ on the path $W_{L}(y)$. Of course, in the case $u=x_{0}$ we set $e_{L}^{-}=e_{-\infty}$. Also, if $u$ is on the right side of $\mathcal{B}$, let $e_{R}^{+}$and $e_{R}^{-}$be, respectively, the edges immediately after and immediately before $u$ on $W_{R}(y)$. Again, if $u=x_{0}$ we set $e_{R}^{-}=e_{-\infty}$. Note that if $u=x$, then $u$ is on both sides of $\mathcal{B}$.

If $u$ is strictly on the left side of $\mathcal{B}$, then the fact that $e$ is in the exterior of $\mathcal{B}$ implies that $e$ is left of $e_{L}^{+}$in the $\left(u, e_{L}^{-}\right)$-ordering. Let $N^{\prime}$ be the concatenation of $W_{L}(u), W$ and $N$. Then $N^{\prime}$ is a witnessing path from $x_{0}$ to $y$ contradicting the fact that $W_{L}(y)$ is leftmost. We conclude that $u$ cannot be strictly on the left side of $\mathcal{B}$. A symmetric argument shows that $u$ cannnot be strictly on the right side of $\mathcal{B}$. We conclude that $u=x$.

We now identify two cases according to the cyclic ordering on the edges $e_{L}^{+}, e_{L}^{-}, e_{R}^{+}, e_{R}^{-}$.

Case 1. $e_{L}^{-} \prec e_{L}^{+} \prec e_{R}^{+} \prec e_{R}^{-} \preccurlyeq e_{L}^{-}$.
We note that this case holds when $x=x_{0}$. Now we claim that $e_{L}^{-} \prec e \prec e_{R}^{-}$. This claim holds trivially if $x=x_{0}$. Now suppose $x \neq x_{0}$ and let $\left(z_{0}, \ldots, z_{m}\right)$ be the sequence of common points of $y$. The assumption in Case 1 holds when $m>0$ and $x=z_{m-1}$ is not a reversing element of $y$. Then $y$ is in the exterior of $\mathcal{B}_{m-1}$ - the block between $z_{m-2}$ and $z_{m-1}$. As noted previously, this requires that all vertices and edges of witnessing paths from $x$ to $y$, except the vertex $x$, are in the exterior of $\mathcal{B}_{m-1}$. This requires the cyclic ordering $e_{L}^{-} \prec e \prec e_{R}^{-}$, as desired.

Since $e$ is not in $\mathcal{B}$, it follows that either $e_{L}^{-} \prec e \prec e_{L}^{+}$, or $e_{R}^{+} \prec e \prec e_{R}^{-}$. If the first option holds, then the concatenation of $W_{L}(x), W$ and $N$ contradicts $W_{L}(y)$ being leftmost. If the second option holds, then the concatenation of $W_{R}(x), W$ and $N$ contradicts $W_{R}(y)$ being rightmost. With these observations, the proof of Case 1 is complete.

Case 2. $\quad e_{L}^{-} \prec e_{R}^{-} \prec e_{L}^{+} \prec e_{R}^{+} \prec e_{L}^{-}$.
We note that in this case $x \neq x_{0}$. Let $\left(z_{0}, \ldots, z_{m}\right)$ be the sequence of common points of $y$. We have $m \geqslant 2$, and $x=z_{m-1}$ is a reversing element of $y$. Thus, $y$ is in the interior of $\mathcal{B}_{m-1}$. Also, all edges and vertices of witnessing paths from $x$ to $y$, except $x$, are in the interior of $\mathcal{B}_{m-1}$. It follows that we have the cyclic ordering $e_{R}^{-} \prec e \prec e_{L}^{-}$. Since $e$ is not in $\mathcal{B}$, it follows that either
$e_{R}^{-} \prec e \prec e_{L}^{+}$or $e_{R}^{+} \prec e \prec e_{L}^{-}$. If the first option holds, then the concatenation of $W_{L}(x), W$ and $N$ contradicts $W_{L}(y)$ being leftmost. If the second option holds, then the concatenation of $W_{R}(x), W$ and $N$ contradicts $W_{R}(y)$ being rightmost. Since the two cases are exhaustive, the proof of the proposition is complete.

Proposition 14. Let $z \in U_{P}\left(x_{0}\right)$, let $j$ be an integer with $0 \leqslant j \leqslant \operatorname{sd}(z)$, let $\mathcal{B}$ be a block of $\operatorname{shad}_{j}(z)$, let $x=\min (\mathcal{B})$ and $y=\max (\mathcal{B})$, and let $u$ be an element of $P$ that belongs to $\mathcal{B}$ but is not the base element of $\operatorname{shad}_{j}(z)$. Then the following statements hold:
(i) $\operatorname{sd}(u) \geqslant j$.
(ii) $\operatorname{shad}_{j}(u) \subseteq \operatorname{shad}_{j}(y)$, with equality holding if and only if $u \geqslant_{P} y$.

Proof. We first prove statement (i). Let $\left(z_{0}, \ldots, z_{m}\right)$ be the sequence of common points of $z$. Note that $x$ is in the sequence, say $x=z_{i}$. Let $z_{i^{\prime}}$ be the base element of the $\operatorname{shad}_{j}(z)$. By our assumption we have $z_{i^{\prime}} \leqslant z_{i}=x \leqslant u$ in $P$ and at least one of the inequalities is strict. Proposition 12 implies that $x$ is also a common point for $u$. Thus, $\left(z_{0}, \ldots, z_{i}\right)$ is a prefix of the sequence of common points of $u$. Of these, $j$ elements are reversing for $z$ so also for $u$ (and in the case $z_{i}=u$, the $j$ reversing elements are before $z_{i}$ as $z_{i^{\prime}}<u$ in $\left.P\right)$. Thus, $\operatorname{sd}(u) \geqslant j$. This completes the proof of statement (i).

Now we prove statement (ii). Any block of $\operatorname{shad}_{j}(z)$ that precedes $\mathcal{B}$ is also a block of both $\operatorname{shad}_{j}(u)$ and $\operatorname{shad}_{j}(y)$. Now consider a block $\mathcal{D}$ of $\operatorname{shad}_{j}(u)$ with $x \leqslant \min (\mathcal{D})$. Then the two sides of $\mathcal{D}$ are witnessing paths from $\min (\mathcal{D})$ to $\max (\mathcal{D})$. However, $x \leqslant \min (\mathcal{D})<\max (\mathcal{D}) \leqslant u$ in $P$. It follows from Proposition 13 that all edges and vertices of the two sides of $\mathcal{D}$ are in $\mathcal{B}$. This implies that $\mathcal{D} \subseteq \mathcal{B} \subseteq \operatorname{shad}_{j}(y)$.

Now suppose that $\operatorname{shad}_{j}(y)=\operatorname{shad}_{j}(u)$. Recall that all elements $w$ of $P$ on the boundary of $\operatorname{shad}_{j}(u)$ satisfy $w \leqslant u$ in $P$. Since $y$ is on the boundary of $\operatorname{shad}_{j}(u)$, we conclude $y \leqslant u$ in $P$. Now suppose that $y \leqslant u$ in $P$. As noted previously, $x \in W_{L}(u)$. By Proposition 13, the whole path $x W_{L}(u) u$ lies in $\mathcal{B}$. Note that the left side of $\mathcal{B}$, i.e. $x W_{L}(y) y$ is the lefmost possible continuation (of $W_{L}(u)$ ) from $x$ that stays in $\mathcal{B}$. Therefore, the assumption $y \leqslant u$ in $P$ implies that $x W_{L}(y) y$ is a part of $W_{L}(y)$. In particular, $y \in W_{L}(u)$. Symmetrically, we argue that $y \in W_{R}(u)$. All this together implies that $\operatorname{shad}_{j}(y)=\operatorname{shad}_{j}(u)$ as desired. With this observation, the proof of statement (ii) is complete.
4.3. The Address of an Inside Pair. Let $(a, b)$ be a pair of distinct elements in $U_{P}\left(x_{0}\right)$. We define the depth of $(a, b)$ to be the least non-negative integer $j$ such that $\operatorname{shad}_{j}(a) \neq \operatorname{shad}_{j}(b)$. Note that if $j$ is the depth of $(a, b)$, then $(1) \operatorname{shad}_{j}(a)$ and $\operatorname{shad}_{j}(b)$ have the same base element; and $(2) \operatorname{shad}_{j}(a)$ and $\operatorname{shad}_{j}(b)$ do not have the same terminal element.

Proposition 15. Let $(a, b) \in \operatorname{Inc}(P)$, and let $j$ be the depth of $(a, b)$. Then $(a, b)$ is an inside pair if and only if $a$ is in the interior of $a$ block of $\operatorname{shad}_{j}(b)$.

Proof. Suppose first that $a$ is in the interior of the block $\mathcal{B}$ of $\operatorname{shad}_{j}(b)$. We show that $(a, b)$ is an inside pair, i.e., we must show (1) $W_{L}(b)$ is $x_{0}$-left of $W_{L}(a)$; and (2) $W_{R}(b)$ is $x_{0}$-right of $W_{R}(a)$. We prove the first of these two statements. The argument for the second is symmetric. Let $x=\min (\mathcal{B})$ and let $y=\max (\mathcal{B})$. Proposition 12 implies that $x$ is in both $W_{L}(a)$ and $W_{R}(a)$. This means that $W_{L}(a)$ and $W_{L}(b)$ coincide from $x_{0}$ to $x$, and also $W_{R}(a)$ and $W_{R}(b)$ coincide from $x_{0}$ to $x$.

Note that if $a \geqslant y$ in $P$, then Proposition 14 implies $\operatorname{shad}_{j}(a)=\operatorname{shad}_{j}(y)$. Since the depth of $(a, b)$ is $j$, we must have $\operatorname{shad}_{j}(a)=\operatorname{shad}_{j}(y) \nsubseteq \operatorname{shad}_{j}(b)$. This means that if $a \geqslant y$ in $P$, then $b$ is not in $\mathcal{B}$ and the edges of $W_{L}(b)$ and $W_{R}(b)$ immediately after $y$ are not in $\mathcal{B}$.

Since $x$ and $a$ are in $\mathcal{B}$, Proposition 13 implies that all edges and vertices of $x W_{L}(a) a$ are in $\mathcal{B}$. Let $z$ be the largest point on the left side of $\mathcal{B}$ such that $z \in W_{L}(a)$. As we discussed $x \leqslant z \leqslant y$ in $P$ and clearly, $x W_{L}(a) z$ is a prefix of the left side of $\mathcal{B}$. Let $e^{+}$and $e^{-}$be the edges of $W_{L}(b)$ that are, respectively, immediately after and immediately before the vertex $z$. Again, if $z=x=x_{0}$ then $e^{-}=e_{-\infty}$. Let $e$ be the edge of $W_{L}(a)$ immediately after $z$. Since $e$ is inside $\mathcal{B}$ and $e^{+}$is on the boundary of $\mathcal{B}$ or outside $\mathcal{B}$ (if $z=y$; by the previous paragraph), we conclude that $e^{+}$is left of $e$ in the $\left(z, e^{-}\right)$-ordering. This implies $W_{L}(b)$ is $x_{0}$-left of $W_{L}(a)$. A symmetric argument shows that $W_{R}(b)$ is $x_{0}$-right of $W_{R}(a)$. Together, these statements imply that $(a, b)$ is an inside pair.

For the second part of the proof, we assume that $(a, b)$ is an inside pair and show that there is some block of $\operatorname{shad}_{j}(b)$ such that $a$ is in the interior of that block. Let $\left(a_{0}, \ldots, a_{m}\right)$ and $\left(b_{0}, \ldots, b_{n}\right)$ be the sequences of common elements of $a$ and $b$, respectively. Since $(a, b)$ is of depth $j, \operatorname{shad}_{j}(a)$ and $\operatorname{shad}_{j}(b)$ have the same base element, which we denote as $x_{j}$. Let $x$ be the last element of the common prefix of the two sequences, say $x=a_{s}=b_{s}$. Note that $x$ does not occur before $x_{j}$ and $x \notin\{a, b\}$. Let $y=b_{s+1}$ be the common point of $b$ that occurs in the sequence immediately after $x$ and let $\mathcal{B}$ be the block of $\operatorname{shad}_{j}(b)$ with $x=\min (\mathcal{B})$ and $y=\max (\mathcal{B})$. Let $e_{L}^{-}, e_{R}^{-}$be the edges of $W_{L}(b)$ and $W_{R}(b)$ respectively that are immediately before $x$ (as usual $e_{L}^{-}=e_{R}^{-}=e_{-\infty}$ if $x=x_{0}$ ). Note that these edges are also in $W_{L}(a)$ and $W_{R}(a)$, respectively. Let $e_{L}^{+}, e_{R}^{+}$be the edges of $W_{L}(b)$ and $W_{R}(b)$, respectively, that are immediately after $x$.

We first consider the case when $x=x_{j}$. In this case, the clockwise cyclic ordering of our four distinguished edges around $x$ is $e_{L}^{-} \preccurlyeq e_{R}^{-} \prec e_{L}^{+} \preccurlyeq e_{R}^{+} \prec e_{L}^{-}$(the first $\preccurlyeq$ becomes $=$ only if $x=x_{j}=x_{0}$ and then both edges coincide with $e_{-\infty}$ ). Now immediately from the fact that $(a, b)$ is an inside pair, we conclude that $e_{L}^{+} \preccurlyeq e \preccurlyeq e_{R}^{+}$when $e$ is either the first edge of $x W_{L}(a) a$ or the first edge of $x W_{R}(a) a$. Therefore both edges belong to $\mathcal{B}$.

We claim that $a$ is in $\mathcal{B}$. In order to prove that, note first that at least one of the paths $x W_{L}(a) a, x W_{R}(a) a$ is not going through $y$. Indeed, if both contain $y$, then $y$ would be a common point of $a$, but it is not. Let $W$ be one of the two paths $x W_{L}(a) a, x W_{R}(a) a$ avoiding $y$. We already proved that the first edge of $W$ is in $\mathcal{B}$. Now in order to get a contradiction suppose that $a$ is not in $\mathcal{B}$. Therefore, $W$ has to eventually leave $\mathcal{B}$. Let $z$ be the first element of $W$ such that the edge $e$ immediately after $z$ on $W$ is not in $\mathcal{B}$. This implies that $z$ is on the boundary of $\mathcal{B}$. Since $z \notin\{x, y\}$, we have $x<z<y$ in $P$. Thus, $z$ is strictly on one of the boundaries of $\mathcal{B}$. If $z$ is on the left boundary, then let $e^{-}$and $e^{+}$be the edges of $W_{L}(b)$, respectively, immediately before and immediately after $z$. Since $e$ is not in $\mathcal{B}$, we have that $e$ is left of $e^{+}$in the $\left(z, e^{-}\right)$-ordering. Now Proposition 6 implies that $W_{L}(a)$ is $x_{0}$-left of $W_{L}(b)$, contradicting that $(a, b)$ is an inside pair. If $z$ is on the right boundary, then we argue symmetrically and obtain that $W_{R}(a)$ is $x_{0}$-right of $W_{R}(b)$, which is also a contradiction.

Now we proceed with the case $x \neq x_{j}$ so $x$ is not a reversing element of $b$. In this case, the clockwise cyclic ordering of our four distinguished edges around $x$ is $e_{L}^{-} \prec e_{L}^{+} \preccurlyeq e_{R}^{+} \prec e_{R}^{-} \preccurlyeq$ $e_{L}^{-}$. Again, immediately from the fact that $(a, b)$ is an inside pair, we conclude that either $e_{R}^{-} \prec e \prec e_{L}^{-}$or $e_{L}^{+} \preccurlyeq e \preccurlyeq e_{R}^{+}$when $e$ is either the first edge of $x W_{L}(a) a$ or the first edge of $x W_{R}(a) a$. Consider first the case that $e_{R}^{-} \prec e \prec e_{L}^{-}$. Let $x^{\prime}$ be the common point of $b$ (and a) just before $x$ in the sequence of the common points. Let $\mathcal{B}^{\prime}$ be the block of $\operatorname{shad}_{j}(b)$ (and
$\left.\operatorname{shad}_{j}(a)\right)$ between $x^{\prime}$ and $x$. In this case, we have that $x$ is a reversing element of $a$ and $a$ is in $\mathcal{B}^{\prime}$, as desired. Now consider the case that $e_{L}^{+} \preccurlyeq e \preccurlyeq e_{R}^{+}$. This means that the first edges of both $x W_{L}(a) a$ and $x W_{R}(a) a$ are both in $\mathcal{B}$. This is exactly the setup of the proof we had when $x=x_{j}$. Thus, we proceed as before and we prove that $a$ has to be in $\mathcal{B}$, as desired. This completes the proof of the proposition.

Using Proposition 15 , we make the following definition. The address of an inside pair $(a, b)$ is the uniquely determined pair $(j, \mathcal{B})$ such that
(i) $j$ is the depth of $(a, b)$; and
(ii) $\mathcal{B}$ is the block of $\operatorname{shad}_{j}(b)$ that contains $a$ in its interior.

Now let $(a, b)$ be an inside pair, let $(j, \mathcal{B})$ be the address of $(a, b)$, and let $y=\max (\mathcal{B})$. Then $j=\operatorname{sd}(y)$, and $y \leqslant b$ in $P$. This implies $a \notin y$ in $P$. If $\mathcal{B}$ is the terminal block of $\operatorname{shad}_{j}(b)$, then $a \| y$ in $P$ (as otherwise $a>y$ in $P$ and this $\operatorname{implies} \operatorname{shad}_{j}(a)=\operatorname{shad}_{j}(y)=\operatorname{shad}_{j}(b)$, which is false). If $\mathcal{B}$ is not the terminal block of $\operatorname{shad}_{j}(b)$, then either $a \| y$ in $P$ or $a>y$ in $P$.

Example 16. We illustrate the notion of addresses using the poset whose cover graph is shown in Figure 5:
(i) The address of $(b, z)$ is $\left(0, \mathcal{B}_{1}\right)$.
(ii) The address of $(h, z)$ is $\left(0, \mathcal{B}_{2}\right)$.
(iii) The address of $(v, z)$ is $\left(1, \mathcal{B}_{4}\right)$.
(iv) The address of $(q, z)$ is $\left(2, \mathcal{B}_{6}\right)$.

In an effort to avoid the pathology displayed by the pairs in Figure 6, we introduce a notion of parity for inside pairs. For each $\theta \in\{0,1\}$, we let $I_{\theta}$ consist of all inside pairs $(a, b)$ such that if $(j, \mathcal{B})$ is the address of $(a, b)$, then $j \equiv \theta \bmod 2$. The set of all inside pairs is partitioned as $I_{0} \sqcup I_{1}$.

We now prove a comprehensive technical lemma that provides structural information about strict alternating cycles in $I_{0}$ and $I_{1}$. The conclusions of this lemma justify the notion of parity for inside pairs.

Lemma 17. Let $\theta \in\{0,1\}$, and let $\left(\left(a_{1}, b_{1}\right), \ldots,\left(a_{k}, b_{k}\right)\right)$ be a strict alternating cycle of pairs from $I_{\theta}$. Then there is a pair $(j, \mathcal{B})$ that is a common address of $\left(a_{\alpha}, b_{\alpha}\right)$ for all $\alpha \in[k]$. Moreover, for all $\alpha \in[k]$ the following statements hold:
(i) $\mathcal{B}$ is the terminal block of $\operatorname{shad}_{j}\left(b_{\alpha}\right)$;
(ii) $a_{\alpha}$ and $b_{\alpha}$ are in the interior of $\mathcal{B}$;
(iii) $a_{\alpha} \| \max (\mathcal{B})$ and $b_{\alpha}>\max (\mathcal{B})$ in $P$;
(iv) $\operatorname{sd}\left(a_{\alpha}\right)=j, \operatorname{sd}\left(b_{\alpha}\right)>j$.

Proof. For each $\alpha \in[k]$, let $j_{\alpha}$ be the depth of the pair $\left(a_{\alpha}, b_{\alpha}\right)$. Set $j=\min \left\{j_{\alpha}: \alpha \in[k]\right\}$. By Proposition 15 , for each $\alpha \in[k]$, we can fix a block $\mathcal{B}_{\alpha}$ of $\operatorname{shad}_{j}\left(b_{\alpha}\right)$ such that $a_{\alpha}$ is in the interior of $\mathcal{B}_{\alpha}$.

The remainder of the argument will be organized within a series of three claims.


Figure 6. $\left(\left(a_{1}, b_{1}\right), \ldots,\left(a_{4}, b_{4}\right)\right)$ is a strict alternating cycle of inside pairs. The common address of $\left(a_{1}, b_{1}\right),\left(a_{3}, b_{3}\right)$, and $\left(a_{4}, b_{4}\right)$ is $(0, \mathcal{B})$. However, the address of $\left(a_{2}, b_{2}\right)$ is $\left(1, \mathcal{B}^{\prime}\right)$. These alternating cycles with pairs of different addresses is what we want to avoid by partitioning $I$ into $I_{0}$ and $I_{1}$. In this figure, the black and green edges are oriented left-to-right in the plane, while the red edges are oriented right-to-left.

Claim 17.1. There is a block $\mathcal{B}$ such that for all $\alpha \in[k]$, the following statements hold:
(i) $\mathcal{B}=\mathcal{B}_{\alpha}$ is the terminal block of $\operatorname{shad}_{j}\left(b_{\alpha}\right)$.
(ii) $a_{\alpha}$ and $b_{\alpha}$ are in the interior of $\mathcal{B}$.
(iii) $b_{\alpha}>\max (\mathcal{B})$ in $P$.

Proof. Let $\alpha \in[k]$. Since $a_{\alpha} \|_{P} b_{\alpha}, a_{\alpha} \leqslant P b_{\alpha+1}$, and $a_{\alpha}$ is in the interior of $\mathcal{B}_{\alpha}$, we assert that $b_{\alpha+1}$ is also in the interior of $\mathcal{B}_{\alpha}$. If this assertion fails, then a witnessing path from $a_{\alpha}$ to $b_{\alpha+1}$ would contain a point $z$ from the boundary of $\mathcal{B}_{\alpha}$. This would force $a_{\alpha}<z \leqslant b_{\alpha}$ in $P$. Clearly, this statement is false. Thus, indeed $b_{\alpha+1}$ is in the interior of $\mathcal{B}_{\alpha}$.

Using Proposition 14, this implies the following inclusion relations:

$$
\operatorname{shad}_{j}\left(b_{\alpha+1}\right) \subseteq \operatorname{shad}_{j}\left(\max \left(\mathcal{B}_{\alpha}\right)\right) \subseteq \operatorname{shad}_{j}\left(b_{\alpha}\right)
$$

Since these inclusions hold for all $\alpha \in[k]$, cyclically, we conclude that for all $\alpha \in[k]$,

$$
\operatorname{shad}_{j}\left(b_{\alpha+1}\right)=\operatorname{shad}_{j}\left(\max \left(\mathcal{B}_{\alpha}\right)\right)=\operatorname{shad}_{j}\left(b_{\alpha}\right)
$$

Since $\operatorname{shad}_{j}\left(b_{\alpha}\right)$ are the same for all $\alpha \in[k]$, their terminal blocks also coincide. Let $\mathcal{B}$ be the common terminal block. The fact that $\operatorname{shad}_{j}\left(\max \left(\mathcal{B}_{\alpha}\right)\right)=\operatorname{shad}_{j}\left(b_{\alpha}\right)$ implies that $\mathcal{B}_{\alpha}$ is the terminal block of $\operatorname{shad}_{j}\left(b_{\alpha}\right)$, so $\mathcal{B}=\mathcal{B}_{\alpha}$ for all $\alpha \in[k]$. The fact that $\operatorname{shad}_{j}\left(b_{\alpha+1}\right)=\operatorname{shad}_{j}(\max (\mathcal{B}))$ coupled with the fact that $b_{\alpha+1}$ is in the interior of $\mathcal{B}$, together with Proposition 14 imply $b_{\alpha+1}>\max (\mathcal{B})$ in $P$. With these observations, the proof of the claim is complete.

Let $y=\max (\mathcal{B})$, and let $M$ be the set of all $\alpha \in[k]$ such that $j_{\alpha}=j$. Since $j=\min \left\{j_{\alpha}: \alpha \in\right.$ $[k]\}$, we know that $M \neq \emptyset$.

Claim 17.2. If $\alpha \in M$, then $a_{\alpha} \|_{P} y$ and $\operatorname{sd}\left(a_{\alpha}\right)=j$.

Proof. Let $\alpha \in M$. Since the depth of $\left(a_{\alpha}, b_{\alpha}\right)$ is $j$, we have $\operatorname{shad}_{j}\left(a_{\alpha}\right) \varsubsetneqq \operatorname{shad}_{j}\left(b_{\alpha}\right)=\operatorname{shad}_{j}(y)$. If $a_{\alpha} \leqslant_{P} y$, then $a_{\alpha} \leqslant_{P} b_{\alpha}$, which is false. If $a_{\alpha}>_{P} y$, then Proposition 14 implies $\operatorname{shad}_{j}\left(a_{\alpha}\right)=$ $\operatorname{shad}_{j}\left(b_{\alpha}\right)$, which is false. We conclude that $a_{\alpha} \|_{P} y$.

Now suppose that $\operatorname{sd}\left(a_{\alpha}\right)>j$. Let $\mathcal{B}^{\prime}$ be the terminal block of $\operatorname{shad}_{j}\left(a_{\alpha}\right)$. Since $a_{\alpha}$ is in the interior of $\mathcal{B}$, and $a_{\alpha} \|_{P} y$, it follows that $\mathcal{B}^{\prime} \varsubsetneqq \mathcal{B}$. In particular, since $w<_{P} a_{\alpha}$ for all elements $w$ on the boundary of $\mathcal{B}^{\prime}$, it follows that $y$ is in the exterior of $\mathcal{B}^{\prime}$. Furthermore, $\operatorname{sd}\left(a_{\alpha}\right)>j$ implies that $a_{\alpha}$ is in the interior of $\mathcal{B}^{\prime}$. Since $a_{\alpha} \leqslant P b_{\alpha+1}$, it follows that $b_{\alpha+1}$ is also in the interior of $\mathcal{B}^{\prime}$.

Now let $W$ be a witnessing path from $y$ to $b_{\alpha+1}$. Then $W$ contains a point $w$ of $P$ that is on the boundary of $\mathcal{B}^{\prime}$. This implies $y<w<a_{\alpha}$ in $P$ so in particular $y<a_{\alpha}$ in $P$, which is false. With this observation, the proof of the claim is complete.

Claim 17.3. $M=[k]$.
Proof. If this claim fails, then after a relabeling of the pairs on the cycle, we can assume that ${ }^{2}$ $j_{1}=j$ and $j_{2} \geqslant j+2$. Let $\mathcal{B}^{\prime}$ be the terminal block of $\operatorname{shad}_{j+1}\left(b_{2}\right)=\operatorname{shad}_{j+1}\left(a_{2}\right)$, and let $\mathcal{B}^{\prime \prime}$ be the block of $\operatorname{shad}{ }_{j+2}\left(b_{2}\right)$ containing $a_{2}$ in its interior. Also, let $x^{\prime}=\min \left(\mathcal{B}^{\prime}\right)$ and $x^{\prime \prime}=\min \left(\mathcal{B}^{\prime \prime}\right)$. Then $\mathcal{B}^{\prime \prime} \varsubsetneqq \mathcal{B}^{\prime}$, and

$$
y \leqslant x^{\prime}<\max \left(\mathcal{B}^{\prime}\right) \leqslant x^{\prime \prime}<\left\{a_{2}, b_{2}\right\} \text { in } P .
$$

If $a_{1} \in \mathcal{B}^{\prime}$, then Proposition 12 implies $y<_{P} x^{\prime} \leqslant_{P} a_{1}$. This implies $y<_{P} a_{1}$, which is false. We conclude that $a_{1} \notin \mathcal{B}^{\prime}$. Noting that $a_{1} \leqslant{ }_{P} b_{2}$, it follows that a witnessing path $W$ from $a_{1}$ to $b_{2}$ must contain a point $z$ from the boundary of $\mathcal{B}^{\prime}$. This implies $a_{1}<z \leqslant \max \left(\mathcal{B}^{\prime}\right) \leqslant x^{\prime \prime}$ in $P$. In turn, this implies $a_{1}<_{P} a_{2}$, which is false. The contradiction completes the proof of the claim.

The four statements of the lemma now follow directly from the three claims, so the proof of Lemma 17 is complete.
4.4. Separating Paths in Shadow Blocks. When $\mathcal{B}$ is a shadow block, we let $x_{\mathcal{B}}=\min (\mathcal{B})$, $y_{\mathcal{B}}=\max (\mathcal{B})$. We have already noted that if $u \in P$ and $u \in \mathcal{B}$, then Proposition 12 implies that $x_{\mathcal{B}}$ belongs to $W_{L}(u)$ and $W_{R}(u)$. In particular, this implies $x_{\mathcal{B}} \leqslant{ }_{P} u$. Now let $u$ be an element of $P$ that is in $\mathcal{B}$. We assign $u$ to $A_{\mathcal{B}}$ if $u \|_{P} y_{\mathcal{B}}$; we assign $u$ to $B_{\mathcal{B}}$ if $u>_{P} y_{\mathcal{B}}$; and we assign $u$ to $Z_{\mathcal{B}}$ if $u \leqslant_{P} y_{\mathcal{B}}$. Evidently, the three sets $A_{\mathcal{B}}, B_{\mathcal{B}}$, and $Z_{\mathcal{B}}$ are pairwise disjoint and they partition the set of elements of $P$ being in $\mathcal{B}$.

Proposition 18. Let $\mathcal{B}$ be a shadow block and let $j=\operatorname{sd}\left(y_{\mathcal{B}}\right)$. Let $a \in A_{\mathcal{B}}, b \in B_{\mathcal{B}}$, and $a \| b$ in $P$. Then $(a, b)$ is an inside pair with address $(j, \mathcal{B})$.

Proof. Since $y_{\mathcal{B}}<b$ in $P$ and $b$ lies in $\mathcal{B}$, we have that $\mathcal{B}$ is the terminal block of $\operatorname{shad}_{j}(b)$. Since $a$ is in the interior of $\mathcal{B}$, by Proposition 15 we get that $(a, b)$ is an inside pair. Recall that $x_{\mathcal{B}}$ is in the sequence of common points of both $a$ and $b$ (see $\operatorname{Proposition~13),~so~} \operatorname{shad}_{i}(a)=\operatorname{shad}_{i}(b)$ for all $i \in\{0, \ldots, j-1\}$. We conclude that the depth of $(a, b)$ is $j$ and therefore the address of $(a, b)$ is $(j, \mathcal{B})$.

[^2]

Figure 7. Left: The red path $N$ from $x_{\mathcal{B}}$ to $y_{\mathcal{B}}$ splits the shadow block $\mathcal{B}$ into regions. The regions marked $L$ are left of $N$, and the regions marked $R$ are right of $N$. Note that each region is a cycle whose boundary consists of two paths, one a subpath of $N$, and the other a subpath of one of the two sides of the shadow block $\mathcal{B}$. Right: We show a separating path $N$ associated with the comparability $a<_{P} b$. The element $p$ is the peak of $N$, and $p \leqslant_{P} b$. When $b \neq p$, the element $b$ can be on either side of $N$.

Let $\mathcal{B}$ be a shadow block, and let $N$ be a path in $G$ (not necessarily a witnessing path) from $x_{\mathcal{B}}$ to $y_{\mathcal{B}}$ such that all edges on $N$ belong to $\mathcal{B}$. When $u$ is either a vertex or edge of $P$ that is in $\mathcal{B}$ (always including the boundary), we can classify $u$ uniquely as being (1) on the path $N$; (2) left of $N$; or (3) right of $N$, using the following scheme. The meaning of the first of these three options is clear. Now suppose that $u$ is not on $N$. We will say that $u$ is left of $N$ if $u$ is in a region in the plane bounded by a cycle formed by two paths, with one path a portion of $N$ and the other path a portion of the left side of $\mathcal{B}$. Symmetrically, we say that $u$ is right of $N$ if $u$ is in a region in the plane bounded by two paths, with one path a portion of $N$ and the other path a portion of the right side of $\mathcal{B}$. We illustrate these conventions on the left side of Figure 7.

Now let $\mathcal{B}$ be a shadow block, and let $(a, b) \in A_{\mathcal{B}} \times B_{\mathcal{B}}$ with $a<_{P} b$. A path $N$ from $x_{\mathcal{B}}$ to $y_{\mathcal{B}}$ in $G$ is a separating path associated with $a<b$ in $P$ if the following statements hold:
(i) $a$ is on $N$ and $x_{\mathcal{B}} N a$ is the suffix of $W_{L}(a)$ starting at $x_{\mathcal{B}}$.
(ii) There is an element $p$ of $B_{\mathcal{B}}$ that is on $N$ such that $p N y_{\mathcal{B}}$ is the part of $W_{L}(b)$ from $y_{\mathcal{B}}$ to $p$ traversed backwards.
(iii) $a<_{P} p$, and $a N p$ is a witnessing path from $a$ to $p$.

The element $p$ referenced in this definition is called the peak of $N$. Note that $z \leqslant p \leqslant b$ in $P$, for every element $z$ of $P$ that is on the path $N$. Note that all edges of $N$ are in $\mathcal{B}$ (by Proposition 13).

Note also that when $\mathcal{B}$ is a shadow block and $(a, b) \in A_{\mathcal{B}} \times B_{\mathcal{B}}$ with $a<b$ in $P$, then there is always a separating path associated with $a<b$ in $P$. The concept of a separating path associated with an inequality $a<_{P} b$ is illustrated on the right side of Figure 7.

Proposition 19. Let $\mathcal{B}$ be a shadow block, let $(a, b) \in A_{\mathcal{B}} \times B_{\mathcal{B}}$ with $a<_{P} b$, and let $N$ be $a$ separating path associated with $a<_{P} b$. If $u$ and $v$ are elements of $P$ that belong to $\mathcal{B}, u<_{P} v$, $u$ is on one side of $N$, and $v$ is on the other side, then $u<_{P} b$.

Proof. Let $W$ be a witnessing path from $u$ to $v$. Proposition 13 implies that all vertices and edges of $W$ are in $\mathcal{B}$. Thus there must be an element $z$ of $P$ common to $W$ and $N$. If $p$ is the peak of $N$, this implies $u<z \leqslant p \leqslant b$ in $P$. Therefore, $u<b$ in $P$.

Proposition 20. Let $\mathcal{B}$ be a shadow block, let $(a, b) \in A_{\mathcal{B}} \times B_{\mathcal{B}}$ with $a<_{P} b$, and let $N$ be $a$ separating path associated with $a<_{P} b$. If $\left(a^{\prime}, b^{\prime}\right) \in A_{\mathcal{B}} \times B_{\mathcal{B}}$, then the following statements hold:
(i) If $\left(a^{\prime}, a\right)$ is a left pair, then $a^{\prime}$ is left of $N$.
(ii) If $\left(a^{\prime}, a\right)$ is a right pair, then $a^{\prime}$ is right of $N$.
(iii) If $\left(b^{\prime}, b\right)$ is a left pair, and $b^{\prime} \|_{P}$ a, then $b^{\prime}$ is right of $N$.
(iv) If $\left(b^{\prime}, b\right)$ is a right pair, and $b^{\prime} \|_{P} a$, then $b^{\prime}$ is left of $N$.

Proof. The arguments for statements (i) and (ii) are symmetric, so we only prove statement (i). Let $w$ be the largest point of $P$ common to $x_{\mathcal{B}} W_{L}\left(a^{\prime}\right) a^{\prime}$ and $x_{\mathcal{B}} W_{L}(a) a$. Since $\left(a^{\prime}, a\right)$ is a left pair, it is in particular an incomparable pair in $P$ (see Proposition 8). Thus, $w<\left\{a, a^{\prime}\right\}$ in $P$. Let $e_{0}$ be the edge common to $W_{L}\left(a^{\prime}\right)$ and $W_{L}(a)$ that is immediately before $w$ (and in the case $w=x_{0}$ let $\left.e_{0}=e_{-\infty}\right)$. Let $e^{\prime}$ and $e$ be the edges immediately after $w$ on $W_{L}\left(a^{\prime}\right)$ and $W_{L}(a)$, respectively. Since $\left(a^{\prime}, a\right)$ is a left pair, we conclude that $e^{\prime}$ is left of $e$ in the ( $w, e_{0}$ )-ordering. Since $e$ is on $N$ and $e_{0}$ is also on $N$ (except the case that $w=x_{\mathcal{B}}$ and $e_{0}$ is not in $\mathcal{B}$ ), we conclude that $e^{\prime}$ is left of $N$.

Now we assume that $a^{\prime}$ is not left of $N$ and argue to a contradiction. This requires that there is a vertex $v$ with $v \neq w$ common to $w W_{L}\left(a^{\prime}\right) a^{\prime}$ and $N$. The choice of $w$ implies $v \notin w N a=w W_{L}(a) a$. Now suppose that $v \in a N p$ where $p$ is the peak of $N$. Then $a<v \leqslant{ }_{P} a^{\prime}$ which is a contradiction as $a \| a^{\prime}$ in $P$. The last possibility is that $v \in p N y_{\mathcal{B}}$. Now we have $y_{\mathcal{B}} \leqslant v \leqslant a^{\prime}$ in $P$, so $y_{\mathcal{B}} \leqslant a^{\prime}$ in $P$, which is contradicts the fact that $a^{\prime} \in A_{\mathcal{B}}$. The contradiction completes the proof of (i).

The proofs of statements (iii) and (iv) are symmetric, so we only prove statement (iii). Let $w$ be the largest element of $P$ common to $y_{\mathcal{B}} W_{L}(b) b$ and $y_{\mathcal{B}} W_{L}\left(b^{\prime}\right) b^{\prime}$. Since $b^{\prime} \|_{P} b$, we must have $w<_{P}\left\{b, b^{\prime}\right\}$. Also $w<p$ in $P$ as otherwise $a<p \leqslant w \leqslant b^{\prime}$ in $P$, which is false. Now we conclude (like in the paragraph before) that the edge of $W_{L}\left(b^{\prime}\right)$ that is immediately after $w$ is right of $N$. Next we assume that $b^{\prime}$ is not right of $N$ and argue to a contradiction. Now we must have an element $v$ of $P$ with $v \neq w$ such that $v$ is common to $w W_{L}\left(b^{\prime}\right) b^{\prime}$ and $N$. The definition of $w$ implies that $v \notin p N y_{\mathcal{B}}$. If $v \in a N p$, then $a \leqslant v \leqslant b^{\prime}$ in $P$. This implies $a<_{P} b^{\prime}$, which is false. It follows that $v \in x_{\mathcal{B}} N a$. Now we have $y_{\mathcal{B}}<v \leqslant a$ in $P$. This implies $y_{\mathcal{B}}<_{P} a$, which is false. The contradiction completes the proof of (iii).

Proposition 21. Let $\mathcal{B}$ be a shadow block, and let $j=\operatorname{sd}\left(y_{\mathcal{B}}\right)$. If $a, a^{\prime} \in A_{\mathcal{B}}$, and $b, b^{\prime} \in B_{\mathcal{B}}$, then the following statements hold:
(i) $b \notin \operatorname{shad}_{j}(a)$.
(ii) $a \notin \operatorname{shad}_{j+1}(b)$.
(iii) If $a \|_{P} a^{\prime}$, and $a \in \operatorname{shad}_{j}\left(a^{\prime}\right)$, then $a \|_{P} b$.
(iv) If $\operatorname{sd}(a)>j$ then $a \|_{P} b$.
(v) If $b \in \operatorname{shad}_{j+1}\left(b^{\prime}\right)$, and $a<_{P} b$, then $a<_{P} b^{\prime}$.
(vi) If $\operatorname{shad}_{j+1}(b)=\operatorname{shad}_{j+1}\left(b^{\prime}\right)$, then $a<_{P} b$ if and only if $a<_{P} b^{\prime}$.

Proof. For the proof of the statement (i), suppose to the contrary that $b \in \operatorname{shad}_{j}(a)$. Since $a \|_{P} y_{\mathcal{B}}$, Proposition 14 implies $y_{\mathcal{B}} \notin \operatorname{shad}_{j}(a)$. Let $W$ be a witnessing path from $y_{\mathcal{B}}$ to $b$. Since $y_{\mathcal{B}} \notin \operatorname{shad}_{j}(a)$ and $b \in \operatorname{shad}_{j}(a)$, the path $W$ must contain an element $z$ of $P$ such that $z$ is also on the boundary of $\operatorname{shad}_{j}(a)$. This implies $y_{\mathcal{B}}<z \leqslant a$ in $P$, and clearly this statement contradicts $a \in A_{\mathcal{B}}$. The contradiction completes the proof of statement (i).

For the proof of statement (ii), suppose to the contrary that $a \in \operatorname{shad}_{j+1}(b)$. Note that $x_{\mathcal{B}} \notin \operatorname{shad}_{j+1}(b)$ and $x_{\mathcal{B}}<_{P} a$. This implies that a witnessing path $W$ from $x_{\mathcal{B}}$ to $a$ contains an element $z$ of $P$ that is also on the boundary of $\operatorname{shad}_{j+1}(b)$. This implies $y_{\mathcal{B}}<z \leqslant a$ in $P$, which contradicts that $a \in A_{\mathcal{B}}$. The contradiction completes the proof of statement (ii).

For the proof of statement (iii), suppose that $a \|_{P} a^{\prime}, a \in \operatorname{shad}_{j}\left(a^{\prime}\right)$, and $a<_{P} b$. By (i) we have $b \notin \operatorname{shad}_{j}\left(a^{\prime}\right)$ and therefore a witnessing path from $a$ to $b$ must contain a point $z$ from the boundary of $\operatorname{shad}_{j}\left(a^{\prime}\right)$. This implies $a<z \leqslant a^{\prime}$ in $P$, which is false. The contradiction completes the proof of statement (iii).

For the proof of statement (iv), suppose that $\operatorname{sd}(a)>j$ and $a<b$ in $P$. Therefore $a$ is in the interior of $\operatorname{shad}_{j}(a)$ and by statement (i), we have $b$ is not in $\operatorname{shad}_{j}(a)$. Let $W$ be a witnessing path from $a$ to $b$. There must be an element $z$ of $W$ that is on the boundary of $\operatorname{shad}_{j}(a)$. This is a clear contradiction as all elements on the boundary of $\operatorname{shad}_{j}(a)$ are below $a$ in $P$.

For the proof of statement (v), suppose that $b \in \operatorname{shad}_{j+1}\left(b^{\prime}\right)$, and $a<_{P} b$. By (ii) we have that $a \notin \operatorname{shad}_{j+1}\left(b^{\prime}\right)$. A witnessing path from $a$ to $b$ must contain an element $z$ of $P$ that is on the boundary of $\operatorname{shad}_{j+1}\left(b^{\prime}\right)$. This implies $a<z \leqslant b^{\prime}$ in $P$. This completes the proof of statement (v).

Statement (vi) follows immediately from (v) as $\operatorname{shad}_{j+1}(b)=\operatorname{shad}_{j+1}\left(b^{\prime}\right)$ implies $b \in$ $\operatorname{shad}_{j+1}\left(b^{\prime}\right)$ and $b^{\prime} \in \operatorname{shad}_{j+1}(b)$.

Proposition 22. Let $\theta \in\{0,1\}$ and let $\left(\left(a_{1}, b_{1}\right), \ldots,\left(a_{k}, b_{k}\right)\right)$ be a strict alternating cycle of pairs from $I_{\theta}$. Let $(j, \mathcal{B})$ be the common address of the pairs on the cycle. If $\alpha$ and $\beta$ are distinct integers in $[k]$, then the following statements hold:
(i) $a_{\alpha} \notin \operatorname{shad}_{j}\left(a_{\beta}\right)$.
(ii) $b_{\alpha} \notin \operatorname{shad}_{j+1}\left(b_{\beta}\right)$.
(iii) $\left(a_{\alpha}, a_{\beta}\right)$ is either a left pair or a right pair.
(iv) $\left(b_{\alpha}, b_{\beta}\right)$ is either a left pair or a right pair.
(v) $\left(a_{\alpha}, a_{\beta}\right)$ is a left pair if and only if $\left(b_{\alpha+1}, b_{\beta+1}\right)$ is a right pair.

Proof. We note that Lemma 17 implies $a_{\alpha} \in A_{\mathcal{B}}$ and $b_{\alpha} \in B_{\mathcal{B}}$, for all $\alpha \in[k]$. Also the strictness of the alternating cycle implies that $\left\{a_{1}, \ldots, a_{k}\right\}$ and $\left\{b_{1}, \ldots, b_{k}\right\}$ are $k$-element antichains in $P$.

For the proof of statement (i), since $a_{\alpha} \| a_{\beta}$, and $a_{\alpha}<b_{\alpha+1}$ in $P$, Proposition 21.(iii) implies $a_{\alpha} \notin \operatorname{shad}_{j}\left(a_{\beta}\right)$.

For the proof of statement (ii), since $a_{\alpha-1}<b_{\alpha}$, and $a_{\alpha-1} \| b_{\beta}$ in $P$, Proposition 21.(v) implies $b_{\alpha} \notin \operatorname{shad}_{j+1}\left(b_{\beta}\right)$.

For the proof of statement (iii), recall that $\left(a_{\alpha}, a_{\beta}\right)$ is either left, or right, or inside, or outside. Recall that the $i$-shadows of $a_{\alpha}$ and $a_{\beta}$ are the same for all $0 \leqslant i<j$, and by statement (i) we have $a_{\alpha} \notin \operatorname{shad}_{j}\left(a_{\beta}\right)$ and $a_{\beta} \notin \operatorname{shad}_{j}\left(a_{\alpha}\right)$. Now Proposition 15 implies that $\left(a_{\alpha}, a_{\beta}\right)$ is not an inside pair and $\left(a_{\beta}, a_{\alpha}\right)$ is not an inside pair. This forces $\left(a_{\alpha}, a_{\beta}\right)$ to be a left pair or a right pair, as desired. Statement (iv) follows along the same lines.

For the proof of statement (v), we may assume, without loss of generality that $\left(a_{\alpha}, a_{\beta}\right)$ is a left pair. Let $N$ be a separating path in the block $\mathcal{B}$ associated with $a_{\beta}<b_{\beta+1}$ in $P$. Then Proposition 20 implies that $a_{\alpha}$ is left of $N$.

We already know that $\left(b_{\alpha+1}, b_{\beta+1}\right)$ is either a left pair or a right pair. Suppose that it is a left pair. Since $b_{\alpha+1} \| a_{\beta}$ in $P$, Proposition 20 implies that $b_{\alpha+1}$ is right of $N$. Now we have $a_{\alpha}$ left of $N, b_{\alpha+1}$ right of $N$, and $a_{\alpha}<b_{\alpha+1}$ in $P$. Proposition 19 now implies $a_{\alpha}<b_{\beta+1}$ in $P$, but this is false. We conclude that $\left(b_{\alpha+1}, b_{\beta+1}\right)$ is a right pair, as desired. With this observation, the proof of statement ( v ) is complete.

The next statement is just a special case of the preceding proposition, but it deserves to be highlighted.

Corollary 23. Let $\theta \in\{0,1\}$ and let $\left(\left(a_{1}, b_{1}\right),\left(a_{2}, b_{2}\right)\right)$ be a strict alternating cycle of inside pairs from $I_{\theta}$. Then $\left(a_{1}, a_{2}\right)$ is a left pair if and only if $\left(b_{1}, b_{2}\right)$ is a left pair.

With these observations, we have reached the end of the common part of the proofs of our two main theorems.

## 5. Boolean Dimension is Bounded

In this section, we give the second part of the proof showing that if $P$ is a poset with a unique minimal element and a planar cover graph, then $\operatorname{bdim}(P) \leqslant 13$. The initial setup is the same as in the last section. We assume that $P$ is a poset with a planar cover graph and a unique minimal denoted $x_{0}$. We fix a plane drawing of the cover graph $G$ of $P$ with $x_{0}$ on the exterior face. Ultimately, we will show that $\operatorname{bdim}(P) \leqslant 13$ by constructing a Boolean realizer $\left(L_{1}, \ldots, L_{13}\right)$ with an appropriate 13 -ary Boolean formula.

Let $\theta \in\{0,1\}$. Recall that $I_{\theta}$ denotes the set of all inside pairs $(a, b)$ such that if $(j, \mathcal{B})$ is the address of $(a, b)$, then $j \equiv \theta \bmod 2$. We then define the following two sets:
(i) $X_{\theta}$ (inside left-safe) consists of all pairs $(a, b) \in I_{\theta}$ for which there does not exist a pair ( $\left.a^{\prime}, b^{\prime}\right) \in I_{\theta}$ such that $a<_{P} b^{\prime}, a^{\prime}<_{P} b$, and $\left(a^{\prime}, a\right)$ is a left pair.
(ii) $X_{\theta}$ (inside right-safe) consists of all pairs $(a, b) \in I_{\theta}$ for which there does not exist a pair $\left(a^{\prime}, b^{\prime}\right) \in I_{\theta}$ such that $a<_{P} b^{\prime}, a^{\prime}<_{P} b$, and $\left(a^{\prime}, a\right)$ is a right pair.

Proposition 24. For each $\theta \in\{0,1\}$, the sets

$$
X_{\theta}(\text { inside left-safe }) \quad \text { and } \quad X_{\theta}(\text { inside right-safe })
$$

are reversible.

Proof. Let $\theta \in\{0,1\}$. We show that $X_{\theta}$ (inside left-safe) is reversible. The argument for $X_{\theta}($ inside right-safe $)$ is symmetric. Suppose to the contrary that $\left(\left(a_{1}, b_{1}\right), \ldots,\left(a_{k}, b_{k}\right)\right)$ is a strict alternating cycle of pairs from $X_{\theta}$ (inside left-safe). Using Lemma 17 , we know that there is a pair $(j, \mathcal{B})$ that is the address of all pairs on the cycle. Also, we know that for all $\alpha \in[k], a_{\alpha} \in A_{\mathcal{B}}$, and $b_{\alpha} \in B_{\mathcal{B}}$. By Proposition 18 it follows that if $\alpha, \beta \in[k]$, and $\beta \neq \alpha+1$, then $\left(a_{\alpha}, b_{\beta}\right)$ is an inside pair whose address is $(j, \mathcal{B})$. In particular, all such pairs are in $I_{\theta}$.

Proposition 22 implies that for distinct $\alpha, \beta \in[k],\left(a_{\alpha}, a_{\beta}\right)$ is either a left pair or a right pair. Since left (right) pairs are transitive (see Proposition 8), we may assume, after a relabeling if necessary, that $\left(a_{\alpha}, a_{1}\right)$ is a left pair, for every $\alpha \in\{2, \ldots, k\}$. Now we know that $\left(a_{k}, b_{2}\right)$ is in $I_{\theta}, a_{1}<b_{2}$ in $P, a_{k}<b_{1}$ in $P$, and ( $a_{k}, a_{1}$ ) is a left pair. This contradicts the assumption that $\left(a_{1}, b_{1}\right)$ is a pair in $X_{\theta}$ (inside left-safe). The contradiction completes the proof.

Propositions 10 and 24 identify the following six sets as being reversible:

$$
\begin{aligned}
& X_{1}=X(\text { left }) \cup X(\text { outside }), \\
& X_{2}=X(\text { right }) \cup X(\text { outside }), \\
& X_{3}=X_{0}(\text { inside left-safe }, \\
& X_{4}=X_{0}(\text { inside right-safe }), \\
& X_{5}=X_{1}(\text { inside left-safe }, \\
& X_{6}=X_{1}(\text { inside right-safe }) .
\end{aligned}
$$

For brevity, we say that a pair $(a, b) \in \operatorname{Inc}(P)$ is dangerous when it does not belong to $X_{1} \cup \cdots \cup X_{6}$. We state for emphasis the following elementary characterization of dangerous pairs.

Proposition 25. Let $(a, b) \in \operatorname{Inc}(P)$. Then $(a, b)$ is dangerous if and only if
(i) $(a, b)$ is an inside pair; and
(ii) if $(a, b)$ has address $(j, \mathcal{B})$ then there are inside pairs $(w, z)$ and $\left(w^{\prime}, z^{\prime}\right)$ also with address ( $j, \mathcal{B}$ ) such that
(a) $a<z, w<b$ in $P,(w, a)$ is a left pair, and $(z, b)$ is a left pair;
(b) $a<z^{\prime}, w^{\prime}<b$ in $P$, and $\left(w^{\prime}, a\right)$ is a right pair, and $\left(z^{\prime}, b\right)$ is also a right pair.

Moroever, $\mathcal{B}$ is a terminal block of $\operatorname{shad}_{j}(b)=\operatorname{shad}_{j}(z)=\operatorname{shad}_{j}\left(z^{\prime}\right)$.
When $(a, b)$ is a dangerous pair with address $(j, \mathcal{B})$, as evidenced by the pairs $(w, z)$ and $\left(w^{\prime}, z^{\prime}\right)$ from Proposition 25, we will refer to $(w, z)$ as a left neighbor of $(a, b)$. Analogously, $\left(w^{\prime}, z^{\prime}\right)$ will be called a right neighbor of $(a, b)$. Note that by Lemma 17, we have $\operatorname{sd}(w)=\operatorname{sd}\left(w^{\prime}\right)=$ $\operatorname{sd}(a)=j$.

For $i \in[6]$, we let $L_{i}$ be a linear extension of $P$ that reverses the incomparable pairs in $X_{i}$ (the linear extensions $L_{1}$ and $L_{2}$ were discussed in the common part of the proof given in the preceding section). Given a pair ( $a, b$ ) of elements of $P$, we can say with certainty that $a \nless b$ in $P$ if there is some $i \in[6]$ such that $a \nless b$ in $L_{i}$. However, if $a \leqslant b$ in $L_{i}$ for all $i \in[6]$, we know that one of the following two statements holds:
(i) $a \leqslant b$ in $P$; or
(ii) $(a, b)$ is a dangerous pair.

The goal for the next seven linear orders $L_{7}, \ldots, L_{13}$ will be be to detect which of these two options applies. These linear orders will not be linear extensions of $P$.

The next step in the proof is an application of a well known technique used by researchers working on Boolean dimension. By convention, we set $\operatorname{sd}\left(x_{0}\right)=-1$. Then for $\theta \in\{0,1\}$, let $M_{\theta}$ denote an arbitrary linear order on the elements $u$ in $P$ such that $\operatorname{sd}(u) \equiv \theta \bmod 2$. Then let $M_{\theta}^{\prime}$ be the dual of $M_{\theta}$. Then set:

$$
L_{7}=M_{0}<M_{1}, \quad L_{8}=M_{0}^{\prime}<M_{1}^{\prime}, \quad \text { and } \quad L_{9}=M_{0}<M_{1}^{\prime}
$$

Now let $a$ and $b$ be distinct elements of $P$. If $a<b$ in $L_{7}$ and $a<b$ in $L_{8}$, then we know $\operatorname{sd}(a)$ is even and $\operatorname{sd}(b)$ is odd. If $a<b$ in $L_{7}$ and $a>b$ in $L_{8}$, then we know $\operatorname{sd}(a)$ and $\operatorname{sd}(b)$ have the same parity. In that case, if $a<b$ in $L_{9}$, then $\operatorname{sd}(a)$ and $\operatorname{sd}(b)$ are both even; while if $a>b$ in $L_{9}$, the $\operatorname{sd}(a)$ and $\operatorname{sd}(b)$ are both odd.

Proposition 26. Let $\mathcal{B}$ be a shadow block, and let $j=\operatorname{sd}\left(y_{\mathcal{B}}\right)$. If $a \in A_{\mathcal{B}}, b, c \in B_{\mathcal{B}}$, and $\operatorname{shad}_{j+1}(b)=\operatorname{shad}_{j+1}(c)$, then $(a, b)$ is dangerous if and only if $(a, c)$ is dangerous.

Proof. Recall, that Proposition 18 states that all incomparable pairs from $A_{\mathcal{B}} \times B_{\mathcal{B}}$ are inside pairs with address $(j, \mathcal{B})$. Let $a \in A_{\mathcal{B}}, b, c \in B_{\mathcal{B}}, \operatorname{shad}_{j+1}(b)=\operatorname{shad}_{j+1}(c)$, and assume that $(a, b)$ is dangerous. Let $(w, z)$ and $\left(w^{\prime}, z^{\prime}\right)$ be, respectively, a left neighbor and a right neighbor of $(a, b)$. Then $a<\left\{z, z^{\prime}\right\}$ in $P, b>\left\{w, w^{\prime}\right\}$ in $P,(w, a)$ is a left pair, and $\left(w^{\prime}, a\right)$ is a right pair. Since $\operatorname{shad}_{j+1}(b)=\operatorname{shad}_{j+1}(c)$, Proposition 21.(vi) implies that if $u \in A_{\mathcal{B}}$, then $u<b$ in $P$ if and only if $u<c$ in $P$. In particular, $a \| b$ in $P$ implies $a \| c$ in $P$, so $(a, c)$ is an inside pair with address $(j, \mathcal{B})$. Also, $b>\left\{w, w^{\prime}\right\}$ in $P$ implies $c>\left\{w, w^{\prime}\right\}$ in $P$. Therefore, the pairs $(w, z)$ and $\left(w^{\prime}, z^{\prime}\right)$ now witness that $(a, c)$ is a dangerous pair. Since this argument can be reversed if we start with the assumption that $(a, c)$ is dangerous, the proof is complete.

For each $\theta \in\{0,1\}$, let $D_{\theta}$ denote the set of dangerous pairs $(a, b)$ such that $(a, b) \in I_{\theta}$. Note that if $(a, b)$ is a dangerous pair with address $(j, \mathcal{B})$, then Lemma 17 implies that $\operatorname{sd}(a)=j$. Therefore, $D_{\theta}$ is exactly the set of all dangerous inside pairs with $\operatorname{sd}(a) \equiv \theta \bmod 2$. The next proposition implies that when treated as a binary relation, the set $D_{\theta}$ is transitive.

Proposition 27. For each $\theta \in\{0,1\}$, if $(a, b),(b, c) \in D_{\theta}$, then $(a, c) \in D_{\theta}$.

Proof. Fix $\theta \in\{0,1\}$ and suppose that $(a, b),(b, c) \in D_{\theta}$. Let $j=\operatorname{sd}(a)$, so $j \equiv \theta \bmod 2$. Since $(a, b)$ is dangerous, we know $\operatorname{sd}(b)>\operatorname{sd}(a)=j$ (see Lemma 17.(iv)). Since $(b, c) \in D_{\theta}$, it follows that $\operatorname{sd}(b) \equiv \theta \bmod 2$ and therefore $\operatorname{sd}(b) \geqslant j+2$. This implies that $b$ and $c$ are in the terminal region of their common $(j+1)$-shadow. Thus $\operatorname{shad}_{j+1}(b)=\operatorname{shad}_{j+1}(c)$ and Proposition 26 implies that $(a, c) \in D_{\theta}$.

Let $a$ and $b$ be elements of $P$. We will then say that the comparability $a<b$ in $P$ tilts right when there exists an element $u$ of $P$ satisfying the following conditions: $(1)(u, a)$ is a right pair; (2) $(u, b)$ is a dangerous pair; and (3) if $(j, \mathcal{B})$ is the address of $(u, b)$, then $a \in A_{\mathcal{B}}$. We note that the fact that $b \in B_{\mathcal{B}}$ coupled with the inequality $a<b$ in $P$ implies (by Proposition 21.(iv)) that $\operatorname{sd}(a)=j$.

Analogously, we will then say that the comparability $a<b$ in $P$ tilts left when there exists an element $v$ of $P$ satisfying the following conditions: (1) $(v, a)$ is a left pair; (2) $(v, b)$ is a dangerous pair; and $(3)$ if $(j, \mathcal{B})$ is the address of $(v, b)$, then $a \in A_{\mathcal{B}}$. Again, we have $\operatorname{sd}(a)=j$.

In the proof of the following proposition, the fact that a dangerous pair has both a left neighbor and a right neighbor is essential.

Proposition 28. No comparability in $P$ tilts both left and right.

Proof. Let $a$ and $b$ be elements of $P$ with $a<b$ in $P$. We assume that the comparability $a<b$ in $P$ tilts both right and left and argue to a contradiction. Let $u$ be an element of $P$ witnessing that $a<b$ in $P$ tilts right and let $v$ be an element of $P$ witnessing that $a<b$ in $P$ tilts left. Therefore, $(u, b)$ and $(v, b)$ are dangerous pairs.

Let $j_{u}$ and $j_{v}$, respectively, be the depth of $(u, b)$ and $(v, b)$. We claim that $j_{u}=j_{v}$. Suppose the opposite, say $j_{u}<j_{v}$. Then consider $\left(j_{v}, \mathcal{B}\right)$ the address of $(v, b)$. Since $(v, b)$ is dangerous, we have that $a \in A_{\mathcal{B}}, b \in B_{\mathcal{B}}$, and $\mathcal{B}$ is a (terminal) block of $\operatorname{shad}_{j_{v}}(b)$. In particular, $\operatorname{shad}_{i}(a)=\operatorname{shad}_{i}(b)$ for all $0 \leqslant i<j_{v}$.

Since $(u, b)$ is an inside pair of depth $j_{u}<j_{v}$, Proposition 15 implies that $u$ is in the interior of $\operatorname{shad}_{j_{u}}(b)=\operatorname{shad}_{j_{u}}(a)$. Recall that $(u, a)$ is a right pair, so in particular an incomparable pair. However, again by Proposition 15, we conclude that $(u, a)$ is an inside pair. This contradicts our assumption that $j_{u}<j_{v}$. Similar contradiction is obtained when we assume $j_{u}>j_{v}$. This proves that $j_{u}=j_{v}$.

Let $j=j_{u}=j_{v}$ and let $(j, \mathcal{B}),\left(j, \mathcal{B}^{\prime}\right)$ be the addresses of $(u, b)$ and $(v, b)$, respectively. By Proposition 25 both $\mathcal{B}$ and $\mathcal{B}^{\prime}$ must be a terminal block of $\operatorname{shad}_{j}(b)$. Thus, $\mathcal{B}=\mathcal{B}^{\prime}$ and $(u, b)$, $(v, b)$ have a common address $(j, \mathcal{B})$.

Let $N$ be a separating path in $\mathcal{B}$ that is associated with the inequality $a<b$ in $P$, and let $p$ be the peak of $N$. Since $(u, a)$ is a right pair, and $(v, a)$ is a left pair, Proposition 20 implies that $u$ is right of $N$, and $v$ is left of $N$.

Let $(w, z)$ be a left neighbor of $(v, b)$, and let $\left(w^{\prime}, z^{\prime}\right)$ be a right neighbor of $(u, b)$.
Let $s$ be the largest element common to $W_{L}(p)$ and $W_{L}(z)$. Clearly, $s$ lies in $y_{\mathcal{B}} N p$. Then $s \neq z$ as otherwise $z=s \leqslant p \leqslant b$ in $P$. Let $e^{-}$be the edge of $W_{L}(z)$ before $s$. Note that either $e^{-}$is in $N$ or $s=y_{\mathcal{B}}$ and then $e^{-}$is on the left side of $\mathcal{B}$. Let $e^{+}$be the edge of $W_{L}(z)$ immediately after $s$. Then $e^{+}$is not on $N$, so we have identified two cases: $e^{+}$is right of $N$ and $e^{+}$is left of $N$.

Case 1. $e^{+}$is right of $N$.
In this case, we are going to show that $z$ is also right of $N$. Suppose the opposite, so $z$ is not left of $N$. This implies that $s W_{L}(z) z$ intersects $N$, say at element $t$ with $t \neq s$. First, if $s=p$ then $p W_{L}(z) t N p$ is a directed cycle in $P$, contradiction. Now, suppose that $s \neq p$ so $s<p$ in $P$ and let $e$ be the first edge of $s N p$. Since $e^{+}$is right of $N$, we have that $e^{+}$is left of $e$ in the $\left(s, e^{-}\right)$-ordering. Now the witnessing path $x_{0} W_{L}(p) s W_{L}(z) t N p$ is $x_{0}$-left of $W_{L}(p)$ which should be the leftmost path from $x_{0}$ to $p$, contradiction.

Therefore, we have that $z$ is right of $N$. Since $v$ is left of $N$, and $v<z$ in $P$, Proposition 19 forces $v<b$ in $P$, which is false. This shows that Case 1. cannot hold.

Case 2. $e^{+}$is left of $N$.

In this case, if we have $s<p$ in $P$, then $e^{+}$is right of $e$ in the $\left(s, e^{-}\right)$-ordering, where $e$ is the first edge of $s N p$. This would imply $W_{L}(z)$ being $x_{0}$-right of $W_{L}(b)$, contradicting that $(z, b)$ is a left pair.

Therefore, $s=p$ and all edges and vertices of $p W_{L}(z) z$, except $p$, are left of $N$. Also, all edges and vertices of $p W_{L}(b) b$, except $p$, are left of $N$.

Let $s^{\prime}$ be the largest element common to $W_{L}(p)$ and $W_{L}\left(z^{\prime}\right)$. Clearly, $s^{\prime}$ lies in $y_{\mathcal{B}} N p$. Note that $s^{\prime} \neq z^{\prime}$ as otherwise $z^{\prime}=s^{\prime} \leqslant b$ in $P$ which is false. Let $e^{\prime-}$ be the edge of $W_{L}\left(z^{\prime}\right)$ before $s^{\prime}$. Note that either $e^{\prime-}$ is in $N$ or $s^{\prime}=y_{\mathcal{B}}$ and then $e^{\prime-}$ is on the left side of $\mathcal{B}$. Let $e^{\prime+}$ be the edge of $W_{L}\left(z^{\prime}\right)$ immediately after $s^{\prime}$. Then $e^{\prime+}$ is not on $N$.

Let $e^{\prime}$ be the first edge of $s^{\prime} W_{L}(b) b$. Now, the fact that $\left(z^{\prime}, b\right)$ is a right pair implies that $W_{L}\left(z^{\prime}\right)$ is $x_{0}$-right of $W_{L}(b)$. This implies that $e^{\prime+}$ is right of $e^{\prime}$ in the $\left(s^{\prime}, e^{\prime-}\right)$-ordering. Since $e^{\prime}$ is on $N$ (if $s^{\prime}<p$ in $P$ ) or $e^{\prime}$ is left of $N$ (if $s^{\prime}=p$ ), we conclude that $e^{++}$must be left of $N$.

Consider now the case that $z^{\prime}$ is left of $N$. Recall that $u$ is right of $N$. Since $u<z^{\prime}$ in $P$, Proposition 19 forces $u<b$ in $P$, which is false. This shows that $z^{\prime}$ is not left of $N$.

Since $e^{\prime+}$ is left of $N$ but $z^{\prime}$ is not left of $N$, there is an element $t^{\prime} \neq s^{\prime}$ in $s W_{L}\left(z^{\prime}\right) z^{\prime}$ such that $t^{\prime}$ is on $N$. Then there is a cycle $\mathcal{D}$ in $G$ formed by $s^{\prime} W_{L}\left(z^{\prime}\right) t^{\prime}$ and $t^{\prime} N p N s^{\prime}$. Note that $b$ and $z$ are in the interior of $\mathcal{D}$.

All elements on the boundary of $\mathcal{D}$ are below $p$ in $P$ so also below $b$ in $P$. It follows that all elements on and inside $\mathcal{D}$ are in $\operatorname{shad}_{j+1}(b)$. In particular, $z \in \operatorname{shad}_{j+1}(b)$. This is a contradiction with Proposition 21.(v) since $((v, b),(w, z))$ is a strict alternating cycle of size 2.

For each $\theta \in\{0,1\}$, we will now define a binary relation $\mathrm{NTR}_{\theta}$ as follows:
(i) we assign to $\operatorname{NTR}_{\theta}$ pairs $\left(x_{0}, u\right)$ for all $u \neq x_{0}$ in $P$; and
(ii) we assign to $\operatorname{NTR}_{\theta}$ all pairs $(a, b)$ such that $a, b \in U_{P}\left(x_{0}\right), \operatorname{sd}(a) \equiv \theta \bmod 2$, and $a<{ }_{P} b$ except those pairs $(a, b)$ where the comparability $a<_{p} b$ tilts right.

Note that the abbreviation NTR in this notation is short for "not tilting right." We also define for each $\theta \in\{0,1\}$ in the obvious symmetric manner a second binary relation denoted $\mathrm{NTL}_{\theta}$ :
(i) we assign to $\mathrm{NTL}_{\theta}$ pairs $\left(x_{0}, u\right)$ for all $u \neq x_{0}$ in $P$; and
(ii) we assign to $\mathrm{NTL}_{\theta}$ all pairs $(a, b)$ such that $a, b \in U_{P}\left(x_{0}\right), \operatorname{sd}(a) \equiv \theta \bmod 2$, and $a<_{P} b$ except those pairs $(a, b)$ where the comparability $a<_{P} b$ tilts left.

Proposition 29. For each $\theta \in\{0,1\}$, the binary relations $\mathrm{NTR}_{\theta}$ and $\mathrm{NTL}_{\theta}$ are strict partial orders, i.e., each of $\mathrm{NTR}_{\theta}$ and $\mathrm{NTL}_{\theta}$ is irreflexive and transitive.

Proof. We give the argument to show that $\mathrm{NTR}_{\theta}$ is a strict partial order. The argument for $\mathrm{NTL}_{\theta}$ is symmetric. We first note that $\mathrm{NTR}_{\theta}$ is irreflexive since $a<_{P} b$ whenever $(a, b) \in$ $\mathrm{NTR}_{\theta}$. It remains only to show that $\mathrm{NTR}_{\theta}$ is transitive.

Let $(a, b),(b, c) \in \operatorname{NTR}_{\theta}$. Then, $a<b<c$ in $P$, so $a<c$ in $P$. It follows that $(a, c) \in \operatorname{NTR}_{\theta}$ unless the comparability $a<_{P} c$ tilts right. We assume that this is the case and argue to a contradiction.

Let $u$ be an element of $P$ witnessing that the comparability $a<_{P} c$ tilts right, and let $(j, \mathcal{B})$ be the address of the dangerous pair $(u, c)$. Then, we know that $a \in A_{\mathcal{B}}, c \in B_{\mathcal{B}}$. Since $a<c$ in $P$, Proposition (iv) implies that $\operatorname{sd}(a)=j$. Since $(a, b) \in \mathrm{NTR}_{\theta}$, we conclude that $j \equiv \theta$ $\bmod 2$. Since $a<_{P} b$ and $a \in A_{\mathcal{B}}$, we know that $b \in A_{\mathcal{B}} \cup B_{\mathcal{B}}$, so we have identified two cases.

Case 1. $b \in A_{\mathcal{B}}$.
We assert that the element $u$ evidences that the comparability $b<_{P} c$ tilts right. Since we already know that $(u, c)$ is dangerous and $b \in A_{\mathcal{B}}$, all we need to show is that $(u, b)$ is a right pair. Since $u \|_{P}\{a, c\}$, and $a<_{P} b<_{P} c$, we know $u \|_{P} b$.

Let $s$ be the largest element of $P$ common to $W_{L}(a)$ and $W_{L}(u)$. Let $e^{-}$and $e^{+}$be the edges of $W_{L}(u)$ immediately before and immediately after $s$. Again, in case $s=x_{0}$ we set $e^{-}=e_{-\infty}$. Let $e=s s^{\prime}$ be the edge of $W_{L}(a)$ immediately after $s$. Since $(u, a)$ is a right pair, $e^{+}$is right of $e$ in the $\left(s, e^{-}\right)$-ordering. Note that $s<s^{\prime} \leqslant a<b$ in $P$. Thus, Proposition 6 implies that $W_{L}(b)$ is $x_{0}$-left of $W_{L}(u)$. Therefore, $(b, u)$ is a left pair or $(b, u)$ is an inside pair. We want to exclude the latter case.

Suppose to the contrary that $(b, u)$ is an inside pair. Then $b \in \operatorname{shad}_{j}(u)$. Since $b \|_{P} u$, Proposition 21.(iii) implies that $b \nless c^{\prime}$ for all $c^{\prime} \in B_{\mathcal{B}}$. This contradicts that $b<c$ in $P$. Thus, $(b, u)$ is not an inside pair. We conclude that $(u, b)$ is a right pair as desired.

This of course contradicts the fact that $(b, c) \in \operatorname{NTR}_{\theta}$.
Case 2. $\quad b \in B_{\mathcal{B}}$.
We assert that the element $u$ evidences that $a<_{p} b$ tilts right. All we need to prove is that $(u, b)$ is a dangerous pair. Note that $b \in B_{\mathcal{B}}$ implies $\operatorname{sd}(b) \geqslant j+1$. Since $(b, c) \in \operatorname{NTR}_{\theta}$, we get $\operatorname{sd}(b) \equiv \theta \bmod 2$ which implies $\operatorname{shad}_{j}(b) \geqslant j+2$. Now $b<c$ in $P$ implies that $\operatorname{shad}_{j+1}(b)=\operatorname{shad}_{j+1}(c)$. This combined with Proposition 26 implies that $(u, b)$ is a dangerous pair. Thus, $a<_{P} b$ tilts right. The contradiction completes the proof.

Note that all four strict orders $\mathrm{NTL}_{0}, \mathrm{NTL}_{1}, \mathrm{NTR}_{0}$, and $\mathrm{NTR}_{1}$ are strict suborders of $P$. Therefore, $D_{0}$ and $D_{1}$ are incomparable pairs in all four of them.

Lemma 30. Let $\theta \in\{0,1\}$. Then $D_{\theta}$ is reversible in $\mathrm{NTR}_{\theta}$ and $D_{\theta}$ is reversible in $\mathrm{NTL}_{\theta}$.

Proof. Fix $\theta \in\{0,1\}$. We show that $D_{\theta}$ is reversible in $\mathrm{NTR}_{\theta}$. The argument for $\mathrm{NTL}_{\theta}$ is symmetric.

Suppose to the contrary that $D_{\theta}$ is not reversible in $\operatorname{NTR}_{\theta}$. Let $\left(\left(a_{1}, b_{1}\right), \ldots,\left(a_{k}, b_{k}\right)\right)$ be an alternating cycle in $\operatorname{NTR}_{\theta}$ with $\left(a_{\alpha}, b_{\alpha}\right) \in D_{\theta}$ for all $\alpha \in[k]$. Therefore, for all $\alpha \in[k]$, either $a_{\alpha}=b_{\alpha+1}$ or $\left(a_{\alpha}, b_{\alpha+1}\right) \in \mathrm{NTR}_{\theta}$. Of all such cycles, we choose one for which $k$ is minimum. Note that this automatically implies that the alternating cycle is strict.

Claim 30.1. There is no $\alpha \in[k]$ such that $\left(a_{\alpha+1}, b_{\alpha}\right) \in D_{\theta}$.

Proof. The claim is clearly true when $k=2$, so suppose $k \geqslant 3$. If the claim fails, then let $\left(a_{\alpha+1}, b_{\alpha}\right) \in D_{\theta}$ and consider

$$
\left(\ldots,\left(a_{\alpha-1}, b_{\alpha-1}\right),\left(a_{\alpha+1}, b_{\alpha}\right),\left(a_{\alpha+2}, b_{\alpha+2}\right), \ldots\right)
$$

Thus, it is an alternating cycle in $\mathrm{NTR}_{\theta}$ of size $k-1$ with all pairs in $D_{\theta}$. This contradicts the assumption that $k$ is minimum possible.

Claim 30.2. For all $\alpha \in[k],\left(a_{\alpha}, b_{\alpha+1}\right) \in \operatorname{NTR}_{\theta}\left(\right.$ so $\left.a_{\alpha} \neq b_{\alpha+1}\right)$.

Proof. If the claim fails, then let $a_{\alpha}=b_{\alpha+1}$. Since $\left(a_{\alpha+1}, b_{\alpha+1}\right) \in D_{\theta}$ and $\left(a_{\alpha}, b_{\alpha}\right) \in D_{\theta}$ and by transitivity of $D_{\theta}$ (see Proposition 27), we conclude $\left(a_{\alpha+1}, b_{\alpha}\right) \in D_{\theta}$. This contradicts Claim 30.1.

For each $\alpha \in[k]$, let $\left(j_{\alpha}, \mathcal{B}_{\alpha}\right)$ be the address of the dangerous pair $\left(a_{\alpha}, b_{\alpha}\right)$. We let $A_{\alpha}=A_{\mathcal{B}_{\alpha}}$ and $B_{\alpha}=B_{\mathcal{B}_{\alpha}}$. Then $a_{\alpha} \in A_{\alpha}$ and $b_{\alpha} \in B_{\alpha}$.

Claim 30.3. For all $\alpha \in[k], j_{\alpha} \leqslant j_{\alpha+1}$.

Proof. Since $a_{\alpha}<b_{\alpha+1}$ in NTR $_{\theta}$ we conclude that $a_{\alpha}<b_{\alpha+1}$ in $P$. This forces $b_{\alpha+1} \in A_{\alpha} \cup B_{\alpha}$. Suppose to the contrary that $j_{\alpha}>j_{\alpha+1}$. Let $j=j_{\alpha+1}$.

Since $j_{\alpha} \equiv j_{\alpha+1} \equiv \theta \bmod 2$, we conclude that $j=j_{\alpha+1} \leqslant j_{\alpha}-2$. Since $b_{\alpha}$ and $b_{\alpha+1}$ are together in $\mathcal{B}_{\alpha}$, we know that $\operatorname{sd}\left(b_{\alpha}\right), \operatorname{sd}\left(b_{\alpha+1}\right) \geqslant j_{\alpha} \geqslant j+2$ and for all $0 \leqslant i \leqslant j+1$, we have $\operatorname{shad}_{i}\left(b_{\alpha}\right)=\operatorname{shad}_{i}\left(b_{\alpha+1}\right)$.

Now consider the block $\mathcal{B}_{\alpha+1}$. Note that $a_{\alpha+1} \in A_{\alpha+1}, b_{\alpha}, b_{\alpha+1} \in B_{\alpha+1}$. Proposition 26 implies that $\left(a_{\alpha+1}, b_{\alpha}\right) \in D_{\theta}$. Again, this contradicts Claim 30.1.

The last claim implies that there is an integer $j$ such that $j_{\alpha}=j$ for all $\alpha \in[k]$. We have already noted that for each $\alpha \in[k]$, we know that $b_{\alpha+1} \in A_{\alpha} \cup B_{\alpha}$.

Claim 30.4. For all $\alpha \in[k], \mathcal{B}_{\alpha+1}=\mathcal{B}_{\alpha}$ if $b_{\alpha+1} \in B_{\alpha}$. Furthermore, $\mathcal{B}_{\alpha+1} \varsubsetneqq \mathcal{B}_{\alpha}$ if $b_{\alpha+1} \in A_{\alpha}$.

Proof. If $b_{\alpha+1} \in B_{\alpha}$, then $\mathcal{B}_{\alpha}$ is the terminal block of the $j$-shadow of $b_{\alpha+1}$. Therefore, $\operatorname{shad}_{j}\left(b_{\alpha}\right)=\operatorname{shad}_{j}\left(b_{\alpha+1}\right)$ and in particular $\mathcal{B}_{\alpha}=\mathcal{B}_{\alpha+1}$.

Now suppose that $b_{\alpha+1} \in A_{\alpha}$. Thus $b_{\alpha+1} \| y_{\mathcal{B}_{\alpha}}$. By Proposition 14, we know that the terminal block of $\operatorname{shad}_{j}\left(b_{\alpha+1}\right)$ is strictly contained in $\mathcal{B}_{\alpha}$, as desired.

The preceding claim implies that there is a shadow block $\mathcal{B}$ such that $\mathcal{B}_{\alpha}=\mathcal{B}$, for all $\alpha \in[k]$. Thus, for all $\alpha \in[k]$, the pair $\left(a_{\alpha}, b_{\alpha}\right)$ has address $(j, \mathcal{B})$ and $a_{\alpha} \in A_{\mathcal{B}}, b_{\alpha} \in B_{\mathcal{B}}$.

Claim 30.5. $\left\{a_{1}, \ldots, a_{k}\right\}$ is an antichain in $P$.

Proof. Suppose to the contrary that $a_{\alpha} \leqslant a_{\beta}$ in $P$ for some $\alpha \neq \beta$. This implies that $a_{\alpha}<b_{\beta+1}$ in $P$. Note that $\alpha \neq \beta+1$ as $a_{\alpha} \| b_{\alpha}$ in $P$.

We assert that $a_{\alpha}<_{P} b_{\beta+1}$ does not tilt right. Suppose to the contrary that it tilts right and let $u$ be the witnessing element. Thus $\left(u, b_{\beta+1}\right)$ is dangerous and $\left(u, a_{\alpha}\right)$ is a right pair. Let $\left(j^{\prime}, \mathcal{B}^{\prime}\right)$ be the address of $\left(u, b_{\beta+1}\right)$. Suppose first that $j^{\prime}<j$. Then $u$ is in the interior of $\operatorname{shad}_{j^{\prime}}\left(b_{\beta+1}\right)=\operatorname{shad}_{j^{\prime}}\left(a_{\alpha}\right)$ but this means that $\left(u, a_{\alpha}\right)$ is an inside pair, a contradiction. Therefore $j^{\prime} \geqslant j$.

Observe now that $u \notin B_{\mathcal{B}}$ as otherwise again ( $u, a_{\alpha}$ ) would be an inside pair which is false. This implies that $j^{\prime}=j$. Therefore $u$ must be in the terminal block of the $j$-shadow of $b_{\beta+1}$, namely $\mathcal{B}$. We conclude that $\left(j^{\prime}, \mathcal{B}^{\prime}\right)=(j, \mathcal{B})$.

In particular, $u \in A_{\mathcal{B}}$. Since $a_{\alpha} \leqslant a_{\beta}<b_{\beta+1}$ in $P, u \| a_{\alpha}$ in $P, u \| b_{\beta+1}$, we conclude that $u \| a_{\beta}$ in $P$. Now we aim to show that $\left(u, a_{\beta}\right)$ is a right pair. Let $s$ be the largest element of $P$ common to $W_{L}\left(a_{\alpha}\right)$ and $W_{L}(u)$. Let $e^{-}$and $e^{+}$be the edges of $W_{L}(u)$ immediately before and immediately after $s$. Again, in case $s=x_{0}$ we set $e^{-}=e_{-\infty}$. Let $e=s s^{\prime}$ be the edge of $W_{L}\left(a_{\alpha}\right)$ immediately after $s$. Since $\left(u, a_{\alpha}\right)$ is a right pair, $e^{+}$is right of $e$ in the $\left(s, e^{-}\right)$-ordering. Note that $s<s^{\prime} \leqslant a_{\alpha}<a_{\beta}$ in $P$. Thus, Proposition 6 implies that $W_{L}(u)$ is $x_{0}$-right of $W_{L}\left(a_{\beta}\right)$. Therefore, $\left(u, a_{\beta}\right)$ is either a right pair or an inside pair. Suppose for a moment that $\left(u, a_{\beta}\right)$ is an inside pair. Then $u \in \operatorname{shad}_{j}\left(a_{\beta}\right)$ and by Proposition 21.(ii) $u$ would be incomparable to all elements in $B_{\alpha}$. This contradicts the fact that ( $u, b_{\beta+1}$ ) is a dangerous pair. We conclude that $\left(u, a_{\beta}\right)$ is a right pair, as desired. This implies that $\left(a_{\beta}, b_{\beta+1}\right)$ tilts right, which is a contradiction.

Therefore, we have shown that $a_{\alpha}<b_{\beta+1}$ in $P$ does not tilt right. However, this now implies that $\left(\left(a_{\beta+1}, b_{\beta+1}\right), \ldots,\left(a_{\alpha}, b_{\alpha}\right)\right)$ is a smaller alternating cycle $($ as $\alpha \neq \beta)$ in $\operatorname{NTR}_{\theta}$. The contradiction completes the proof of the claim.

Since each element of $\left\{a_{1}, \ldots, a_{k}\right\}$ is comparable with an element of $B_{\mathcal{B}}$, there cannot be distinct integers $\alpha, \beta \in[k]$ such that $\left(a_{\alpha}, a_{\beta}\right)$ is an inside pair. After a relabeling we may assume that $\left(a_{\alpha}, a_{1}\right)$ is a right pair in $P$ for every $\alpha \in[k]$ with $\alpha \geqslant 2$. However, this implies that the comparability $a_{1}<b_{2}$ in $P$ tilts right as is witnessed by a dangerous pair $\left(a_{2}, b_{2}\right)$. The contradiction completes the proof of the lemma.

We use the preceding lemma to define the remaining linear orders:

$$
\begin{array}{ll}
L_{10} & \text { a linear extension of } \mathrm{NTR}_{0} \text { reversing } D_{0} \\
L_{11} & \text { a linear extension of } \mathrm{NTL}_{0} \text { reversing } D_{0} \\
L_{12} & \text { a linear extension of } \mathrm{NTR}_{1} \text { reversing } D_{1} \\
L_{13} & \text { a linear extension of } \mathrm{NTL}_{1} \text { reversing } D_{1} .
\end{array}
$$

To complete the proof of Theorem 1, we will explain why the linear orders $\left(L_{1}, \ldots, L_{13}\right)$ constitute a Boolean realizer of $P$. The Boolean formula will be clear from the explanation.

Let $a$ and $b$ be two elements of $P$.
(i) If $a \nless b$ in $L_{i}$ for any $i \in[6]$, we know that $a \nless b$ in $P$, and we output 0 .
(ii) Otherwise, we assume that $a \leqslant b$ in $L_{i}$ for all $i \in[6]$. Then we know that either $a \leqslant b$ in $P$ or $(a, b)$ is a dangerous pair. From the responses to the queries as to whether $a \leqslant b$ in $L_{i}$ for $i=7,8,9$, we determine the parity of $\operatorname{sd}(a)$ except the case $a=b$.
(iii) If $\operatorname{sd}(a) \equiv 0 \bmod 2$, and $a \nless b$ in both $L_{10}$ and $L_{11}$, then we now know $(a, b) \in D_{0}$, so $a \nless b$ in $P$ and we output 0 . Indeed, Proposition 28 implies that if $a \leqslant b$ in $P$ then $a \leqslant b$ in at least one of $\mathrm{NTR}_{0}, \mathrm{NTL}_{0}$, and this forces $a \leqslant b$ in at least one of $L_{10}$ and $L_{11}$.
(iv) If $\operatorname{sd}(a) \equiv 1 \bmod 2$, and $a \nless b$ in both $L_{12}$ and $L_{13}$, then we now know $(a, b) \in D_{1}$, so $a \nless b$ in $P$ and we output 0 . Indeed, Proposition 28 implies that if $a \leqslant b$ in $P$ then $a \leqslant b$ in at least one of $\mathrm{NTR}_{1}, \mathrm{NTL}_{1}$, and this forces $a \leqslant b$ in at least one of $L_{12}$ and $L_{13}$.
(v) Otherwise, we know that either $a=b$ or $a \neq b$ and $(a, b)$ is not a dangerous pair. Together with what we gathered so far, we know that $a \leqslant b$ in $P$. Therefore, we output 1.

This completes the proof of Theorem 1.

## 6. Dimension and Standard Example Number

In this section, we give the second part of the proof of Theorem 2, i.e., we show that if $P$ is a poset with a unique minimal element and a planar cover graph, then $\operatorname{dim}(P) \leqslant 2 \operatorname{se}(P)+2$. The initial set up is the same as in the last two sections. We assume that $P$ is a poset with a planar cover graph and a unique minimal element denoted $x_{0}$. We fix a plane drawing of the cover graph $G$ of $P$ with $x_{0}$ on the exterior face.

In Section 4, we showed that there are 2 reversible subsets of $\operatorname{Inc}(P)$ that cover all incomparable pairs in $P$ except the inside pairs. Recall that for $\theta \in\{0,1\}$, we let $I_{\theta}$ consist of all inside pairs $(a, b)$ such that if $(j, \mathcal{B})$ is the address of $(a, b)$, then $j \equiv \theta \bmod 2$. Then we have the partition $I_{0} \sqcup I_{1}$ of the set of all inside pairs. To complete the proof of Theorem 2, we show that for each $\theta \in\{0,1\}$, the set $I_{\theta}$ can be covered with $\operatorname{se}(P)$ reversible sets.

We need a preliminary result that is in the spirit of Proposition 20.
Proposition 31. Let $\theta \in\{0,1\}$, and let $\left((a, b),\left(a^{\prime}, b^{\prime}\right)\right)$ be a strict alternating cycle of pairs from $I_{\theta}$ with $\left(a, a^{\prime}\right)$ a left pair. Let $(j, \mathcal{B})$ be the common address of $(a, b)$ and $\left(a^{\prime}, b^{\prime}\right)$, and let $q$ be an element of $B_{\mathcal{B}}$ such that $(q, b)$ is a left pair. If $N$ is separating path associated with $a<_{P} b^{\prime}$, then $q$ is right of $N$.

Proof. From Proposition 22, we know that $\left(b, b^{\prime}\right)$ is a left pair. Let $p$ be the peak of $N$, and let $z$ be the largest element of $P$ common to $W_{L}(q)$ and $W_{L}(p)$. Clearly, $y_{\mathcal{B}} \leqslant z \leqslant p$. Note that $z \in W_{L}(b)$ as $(q, b)$ and $\left(b, b^{\prime}\right)$ are left pairs. This implies $z<p$ in $P$ as $z=p$ would give $a \leqslant p=z \leqslant b$ in $P$. Let $e^{-}, e^{+}$be the edges in $W_{L}(q)$ immediately before and immediately after $z$. Let $e$ be the edge of $W_{L}(p)$ immediately after $z$. Since $(q, b)$ is a left pair, we have that $e^{+}$is left of $e$ in the $\left(z, e^{-}\right)$-ordering. Since $e$ lies in $N$, we conclude that $e^{+}$is right of $N$. If the element $q$ is not right of $N$, then $z W_{L}(q) q$ must contain an element $w$ from $N$ with $z \neq w$. Since $W_{L}(p)$ and $W_{L}(q)$ are $x_{0}$-consistent, the element $w$ does not belong to $W_{L}(p)$. Therefore, it belongs to $x_{\mathcal{B}} N p$ with $w \neq p$. Now, the witnessing path $x_{0} W_{L}(w) w N p$ provides an alternative witnessing path from $x_{0}$ to $p$ contradicting the property that $W_{L}(p)$ is the leftmost. The contradiction completes the proof.

For each $\theta \in\{0,1\}$, we define a directed graph $H_{\theta}$ whose vertex set is $I_{\theta}$. In $H_{\theta}$, there are two kinds of edges. The first kind will be called a strong edge. When $(a, b)$ and $(b, c)$ are vertices from $I_{\theta}$, we have a strong edge in $H_{\theta}$ from $(a, b)$ to $\left(a^{\prime}, b^{\prime}\right)$ when $a<b^{\prime}$ in $P, a^{\prime}<b$ in $P$, and $\left(a, a^{\prime}\right)$ is a left pair. We then have a weak edge from $(a, b)$ to $\left(a^{\prime}, b^{\prime}\right)$ in $H_{\theta}$ when there is a pair $(u, v) \in I_{\theta}$ such that $\left((a, b),\left(a^{\prime}, b^{\prime}\right),(u, v)\right)$ is a strict alternating cycle, $\left(a, a^{\prime}\right)$ is a left pair, and $\left(a^{\prime}, u\right)$ is a left pair. In discussing weak edges, we will refer to the pair $(u, v)$ as a helper pair for edge $\left((a, b),\left(a^{\prime}, b^{\prime}\right)\right)$.

When $n \geqslant 1$, we will say that a sequence $\left(\left(a_{1}, b_{1}\right), \ldots,\left(a_{n}, b_{n}\right)\right)$ is a directed path in $H_{\theta}$ of size $n$ when there is a directed edge from $\left(a_{i}, b_{i}\right)$ to $\left(a_{i+1}, b_{i+1}\right)$ whenever $1 \leqslant i<n$. Also, we will say that this directed path starts at $\left(a_{1}, b_{1}\right)$. Note that when $n \geqslant 2$, and $1 \leqslant i<n$, $\left(a_{i}, a_{i+1}\right)$ will be a left pair, independent of whether the edge from $\left(a_{i}, b_{i}\right)$ to $\left(a_{i+1}, b_{i+1}\right)$ is strong or weak.

When $(a, b) \in I_{\theta}$, we consider a trivial sequence $((a, b))$ as a directed path of size 1 . For a positive integer $m$, we let $I_{\theta}(m)$ consist of all pairs $(a, b) \in I_{\theta}$ such that the maximum size of a directed path in $H_{\theta}$ starting at $(a, b)$ is $m$.

Lemma 32. For each $\theta \in\{0,1\}$ and each positive integer $m$, the set $I_{\theta}(m)$ is reversible.

Proof. Fix $\theta \in\{0,1\}$ and an integer $m \geqslant 1$. We then assume that $\left(\left(a_{1}, b_{1}\right), \ldots,\left(a_{k}, b_{k}\right)\right)$ is a strict alternating cycle of pairs from $I_{\theta}(m)$ and argue to a contradiction.

Using Lemma 17 , we know that there is a pair $(j, \mathcal{B})$ that is the common address of all the pairs on the cycle. We also know that for every $\alpha \in[k], a_{\alpha} \in A_{\mathcal{B}}$ and $b_{\alpha} \in B_{\mathcal{B}}$. It follows by Proposition 18 that if $\alpha, \beta \in[k]$, and $\beta \neq \alpha+1$, then $\left(a_{\alpha}, b_{\beta}\right)$ is a pair in $I_{\theta}$ and it has address $(j, \mathcal{B})$.

Proposition 22 implies that $\left(a_{\alpha}, a_{\beta}\right)$ is a left pair or a right pair for all $\alpha \neq \beta$ in $[k]$. Therefore we can assume that $a_{1}$ is the lefmost, i.e. $\left(a_{1}, a_{\alpha}\right)$ is a left pair, whenever $2 \leqslant \alpha \leqslant k$. Since $\left(a_{k}, b_{2}\right) \in I_{\theta}$, we have a strong edge from $\left(a_{1}, b_{1}\right)$ to $\left(a_{k}, b_{2}\right)$ in $H_{\theta}$. Therefore, $m \geqslant 2$.

Now suppose $k=2$. Then there is a strong directed edge in $H_{\theta}$ from $\left(a_{1}, b_{1}\right)$ to $\left(a_{2}, b_{2}\right)$. Furthermore, a directed path of size $m$ starting at $\left(a_{2}, b_{2}\right)$ can be extended to a directed path of size $m+1$ starting at ( $a_{1}, b_{1}$ ) simply by prepending $\left(a_{1}, b_{1}\right)$ at the beginning. This implies that $\left(a_{1}, b_{1}\right) \notin I_{\theta}(m)$. The contradiction shows that $k \geqslant 3$.

Consider the 3 -element antichain $\left\{a_{1}, a_{2}, a_{k}\right\}$. We know that $\left(a_{1}, a_{2}\right)$ and ( $a_{1}, a_{k}$ ) are left pairs. We also know that one of ( $a_{2}, a_{k}$ ) and ( $a_{k}, a_{2}$ ) is a left pair.

Case 1. $\left(a_{2}, a_{k}\right)$ is a left pair.
Then consider the strict alternating cycle $\left(\left(a_{1}, b_{1}\right),\left(a_{2}, b_{2}\right),\left(a_{k}, b_{3}\right)\right)$, noting that $\left(a_{1}, a_{2}\right)$ and $\left(a_{2}, a_{k}\right)$ are left pairs. It follows that there is a weak directed edge in $H_{\theta}$ from $\left(a_{1}, b_{1}\right)$ to $\left(a_{2}, b_{2}\right)$, with the pair $\left(a_{k}, b_{3}\right)$ being the helper pair. This implies that a directed path of size $m$ starting at ( $a_{2}, b_{2}$ ) can be extended to a directed path of size $m+1$ starting at ( $a_{1}, b_{1}$ ) simply by prepending $\left(a_{1}, b_{1}\right)$ at the front. This implies $\left(a_{1}, b_{1}\right) \notin I_{\theta}(m)$, a contradiction.

Case 2. $\left(a_{k}, a_{2}\right)$ is a left pair.
It follows from Proposition 22 that $\left(b_{3}, b_{1}\right)$ and $\left(b_{1}, b_{2}\right)$ are left pairs. Let $\left(\left(w_{1}, z_{1}\right), \ldots,\left(w_{m}, z_{m}\right)\right)$ be a directed path of size $m$ in $H_{\theta}$ with $\left(w_{1}, z_{1}\right)=\left(a_{2}, b_{2}\right)$. Then consider the sequence

$$
\left(\left(a_{1}, b_{1}\right),\left(a_{k}, b_{2}\right),\left(w_{2}, z_{2}\right), \ldots,\left(w_{m}, z_{m}\right)\right)
$$

of pairs from $I_{\theta}$. (Recall that $m \geqslant 2$ so ( $w_{2}, z_{2}$ ) exists.) Clearly, this sequence has size $m+1$, and it starts with $\left(a_{1}, b_{1}\right)$. We make the following observations: (1) there is a strong directed edge in $H_{\theta}$ from $\left(a_{1}, b_{1}\right)$ to $\left(a_{k}, b_{2}\right)$; and (2) if $2 \leqslant i<m$, there is a directed edge in $H_{\theta}$ from $\left(w_{i}, z_{i}\right)$ to ( $w_{i+1}, z_{i+1}$ ). Accordingly, the only edge missing so far in $H_{\theta}$ to complete the directed path on our sequence is from $\left(a_{k}, b_{2}\right)$ to $\left(w_{2}, z_{2}\right)$.

Recall that $\left(a_{k}, a_{2}\right)$ is a left pair and $w_{2}<z_{1}=b_{2}$ in $P$. Thus, if $a_{k}<b_{2}$ in $P$ then we will complete a strong edge from $\left(a_{k}, b_{2}\right)$ to $\left(w_{2}, z_{2}\right)$.

We split the argument again depending on the type of edge in $H_{\theta}$ from $\left(w_{1}, z_{1}\right)$ to $\left(w_{2}, z_{2}\right)$.

First, we assume that $\left(\left(w_{1}, z_{1}\right),\left(w_{2}, z_{2}\right)\right)$ is strong. Let $N$ be a separating path in the shadow block $\mathcal{B}$ associated with the comparability $a_{2}=w_{1}<_{P} z_{2}$. Since $\left(a_{k}, a_{2}\right)$ is a left pair, Proposition 20 implies $a_{k}$ is left of $N$. We know that $\left(b_{1}, b_{2}\right)$ is a right pair, $b_{2}=z_{1}$, and $\left(z_{1}, z_{2}\right)$ is a right pair. Thus $\left(b_{1}, z_{2}\right)$ is a right pair. Since $b_{1} \| a_{2}$ in $P$, Proposition 20 implies that $b_{1}$ is right of $N$.

For the comparability $a_{k}<b_{1}$ in $P$, we now have $a_{k}$ left of $N$, and $b_{1}$ right of $N$. Proposition 19 then implies that $a_{k}<b_{2}$ in $P$. This comparability, completes requirements for a strong directed edge from $\left(a_{k}, b_{2}\right)$ to $\left(w_{2}, z_{2}\right)$ in $H_{\theta}$. Therefore, we have completed a directed path in $H_{\theta}$ from $\left(a_{1}, b_{1}\right)$ of length $m+1$. This contradicts the fact that $\left(a_{1}, b_{1}\right) \in I_{\theta}(m)$.

Now, we assume that $\left(\left(w_{1}, z_{1}\right),\left(w_{2}, z_{2}\right)\right)$ is weak.
Then let $\left(\left(w_{1}, z_{1}\right),\left(w_{2}, z_{2}\right),(u, v)\right)$ be a strict alternating cycle evidencing that this weak edge in $H_{\theta}$. Then, again by Lemma $17,(u, v)$ is an inside pair whose address is $(j, \mathcal{B}), u \in A_{\mathcal{B}}$, and $v \in B_{\mathcal{B}}$. Also, $\left(w_{1}, w_{2}\right)$ and $\left(w_{2}, u\right)$ are left pairs.

We note that $\left(b_{3}, b_{1}\right)$ and $\left(b_{1}, b_{2}\right)$ are left pairs. Therefore, $\left(b_{3}, b_{2}\right)$ is a left pair. We apply Proposition 31 for the strict alternating cycle $\left(\left(w_{2}, z_{1}\right),(u, v)\right)$, the separating path $N$ associated with the comparability $w_{2}<_{P} v$, and the element $b_{3}$. The proposition implies that $b_{3}$ is right of $N$. On the other hand, since $\left(a_{2}, w_{2}\right)=\left(w_{1}, w_{2}\right)$ is a left pair, Proposition 20 implies $a_{2}$ is left of $N$. Using Proposition 19 for the comparability $a_{2}<_{P} b_{3}$, we conclude that $w_{1}=a_{2}<_{P} v$, which is false. The contradiction completes the proof.

Proposition 33. Let $\theta \in\{0,1\}$, and let $\left(\left(a_{1}, b_{1}\right),\left(a_{2}, b_{2}\right)\right)$, $\left(\left(a_{2}, b_{2}\right),\left(a_{3}, b_{3}\right)\right)$ be strong edges in $H_{\theta}$. Then $\left(\left(a_{1}, b_{1}\right),\left(a_{3}, b_{3}\right)\right)$ is a strong edge in $H_{\theta}$ as well.

Proof. Since $\left(a_{1}, a_{2}\right)$ and $\left(a_{2}, a_{3}\right)$ are left pairs, we know that $\left(a_{1}, a_{3}\right)$ is a left pair. It suffices to show that $a_{1}<b_{3}$ in $P$ and $a_{3}<b_{1}$ in $P$. We give the argument to show that $a_{1}<b_{3}$ in $P$. The argument for $a_{3}<b_{1}$ in $P$ is symmetric.

By Lemma 17, we can fix $(j, \mathcal{B})$ to be the common address of the three pairs on the path. Then let $N$ be a separating path in the shadow block $\mathcal{B}$ associated with the comparabilty $a_{2}<b_{3}$ in $P$. Since $\left(a_{1}, a_{2}\right)$ is a left pair, Proposition 20 implies that $a_{1}$ is left of $N$. Since $\left(b_{2}, b_{3}\right)$ is a left pair, and $b_{2} \| a_{2}$ in $P$, Proposition 20 also implies that $b_{2}$ is right of $N$. Since $a_{1}$ is left of $N, b_{2}$ is right of $N$, and $a_{1}<b_{2}$ in $P$, Proposition 19 implies $a_{1}<b_{3}$ in $P$. With this observation, the proof is complete.

Let $n \geqslant 2$, and let $\left(\left(a_{1}, b_{1}\right), \ldots,\left(a_{n}, b_{n}\right)\right)$ be a directed path in $H_{\theta}$ with each edge $\left(\left(a_{i}, b_{i}\right),\left(a_{i+1}, b_{i+1}\right)\right)$ being strong for $i \in[n-1]$. Proposition 33 implies that the points in $\left\{a_{1}, \ldots, a_{n}, b_{1}, \ldots, b_{n}\right\}$ determine a copy of the standard example $S_{n}$.

Now we are missing just one final piece to complete the proof.
Proposition 34. For each $\theta \in\{0,1\}, H_{\theta}$ has no directed path on more than $\operatorname{se}(P)$ vertices.

Proof. Let $\theta \in\{0,1\}$ and let $\left(\left(a_{1}, b_{1}\right), \ldots,\left(a_{n}, b_{n}\right)\right)$ be a directed path in $H_{\theta}$. Let $(j, \mathcal{B})$ be the common address of all pairs (and helper pairs) in the path (it exists by Lemma 17). We show that $H_{\theta}$ contains a directed path of the same length with all edges being strong. This will determine a copy of $S_{n}$ in $P$. Therefore, $n \leqslant \operatorname{se}(P)$, as desired.

If the directed edge from $\left(a_{1}, b_{1}\right)$ to $\left(a_{2}, b_{2}\right)$ is strong, we set $\left(u_{1}, v_{1}\right)=\left(a_{1}, b_{1}\right)$ and $\left(u_{2}, v_{2}\right)=$ $\left(a_{2}, b_{2}\right)$. If the directed edge from $\left(a_{1}, b_{1}\right)$ to $\left(a_{2}, b_{2}\right)$ is weak, and is evidenced by the strict alternating cycle $\left(\left(a_{1}, b_{1}\right),\left(a_{2}, b_{2}\right),(u, v)\right)$, we set $\left(u_{1}, v_{1}\right)=\left(a_{1}, v\right)$ and $\left(u_{2}, v_{2}\right)=\left(a_{2}, b_{2}\right)$. Recall that $a_{1} \in A_{\mathcal{B}}$ and $v \in B_{\mathcal{B}}$, therefore by Proposition $18\left(a_{1}, v\right)$ is a vertex in $H_{\theta}$.

In both cases, the following statements hold when $s=2$ :
(i) $\left(u_{s}, v_{s}\right)=\left(a_{s}, b_{s}\right)$.
(ii) $\left(\left(u_{1}, v_{1}\right), \ldots,\left(u_{s}, v_{s}\right)\right)$ is a strong directed path of size $s$ in $H_{\theta}$.

Now we continue the construction as long as $2 \leqslant s<n$ and we will keep two items above as invariants.

If the directed edge from $\left(a_{s}, b_{s}\right)$ to $\left(a_{s+1}, b_{s+1}\right)$ is strong, we simply set $\left(u_{s+1}, v_{s+1}\right)=$ $\left(a_{s+1}, b_{s+1}\right)$ and continue.

Now suppose the directed edge from $\left(a_{s}, b_{s}\right)$ to $\left(a_{s+1}, b_{s+1}\right)$ is weak and is evidenced by the strict alternating cycle $\left(\left(a_{s}, b_{s}\right),\left(a_{s+1}, b_{s+1}\right),(u, v)\right)$ with $\left(a_{s}, a_{s+1}\right)$ and ( $a_{s+1}, u$ ) both being left pairs. Then we set $\left(u_{s+1}, v_{s+1}\right)=\left(a_{s+1}, b_{s+1}\right)$, so that the first statement of our inductive hypothesis is satisfied. However, to maintain the second statement, if $s \geqslant 3$ then we update the choice made for the pair $\left(u_{s}, v_{s}\right)$, i.e. $\left(u_{s}, v_{s}\right)=\left(a_{s}, v\right)$.

To complete the proof, we need only to show that the two requirements are satisfied by the updated path. The first requirement, i.e., $\left(u_{s+1}, v_{s+1}\right)=\left(a_{s+1}, b_{s+1}\right)$ is satisfied trivially. For the second, we note that there is a strong directed edge from $\left(a_{s}, v\right)$ to $\left(a_{s+1}, b_{s+1}\right)$. It suffices to show that if $s \geqslant 3$, there is a strong edge from $\left(u_{s-1}, v_{s-1}\right)$ to $\left(a_{s}, v\right)$.

We know ( $u_{s-1}, a_{s}$ ) is a left pair. We also know that $a_{s}<v_{s-1}$ in $P$. It suffices to show that $u_{s-1}<v$ in $P$. Let $N$ be a separating path in $\mathcal{B}$ associated with the comparability $u_{s+1}<v$ in $P$. Since $\left(u_{s-1}, u_{s+1}\right)$ is a left pair, Proposition 20 implies $u_{s-1}$ is left of $N$. Since $\left(v_{s}, v\right)$ is a left pair, and $v_{s} \| u_{s+1}$ in $P$, Proposition 20 implies that $v_{s}$ is right of $N$. Since $u_{s-1}<v_{s}$ in $P$, Proposition 19 implies $u_{s-1}<v$ in $P$. With this observation, the proof that the inductive construction can continue.

We have now shown that for each $\theta \in\{0,1\}$ and each $m \geqslant 1$, the set $I_{\theta}(m)$ is reversible. We have also shown that $I_{\theta}(m)=\emptyset$ when $m>\operatorname{se}(P)$.

Together, these statements complete the proof of Theorem 2, i.e. the upper bound $\operatorname{dim}(P) \leqslant$ $2 \operatorname{se}(P)+2$ when $P$ is a poset with a planar cover graph and a unique minimal element.

For future reference, we state the following corollary.
Corollary 35. Let $P$ be a poset with a planar cover graph and let $x_{0}$ be a unique minimal element in P. Fix a planar drawing of the cover graph of $P$ with $x_{0}$ on the exterior face.

For all positive integers $k$, if $\operatorname{dim}(P) \geqslant 2 k+1$, then there is a shadow block $\mathcal{B}$ in $P$ with $k$ incomparable pairs $\left(a_{1}, b_{1}\right), \ldots,\left(a_{k}, b_{k}\right)$ such that
(i) $\left\{\left(a_{1}, b_{1}\right), \ldots,\left(a_{k}, b_{k}\right)\right\}$ induce a standard example $S_{k}$ in $P$.
(ii) $a_{i} \| y_{\mathcal{B}}$ in $P$ and $b_{i}>y_{\mathcal{B}}$ in $P$, for all $i \in[k]$.
(iii) $\left(a_{i}, a_{j}\right)$ is a left pair and $\left(b_{i}, b_{j}\right)$ is a left pair for all $i, j \in[k]$ with $i<j$.

## References

[1] C. Biró, P. Hamburger, H. A. Kierstead, A. Pór, W. T. Trotter, and R. Wang. Random bipartite posets and extremal problems. Acta Mathematica Hungarica, 161:618-646, 2020. arXiv:2003.07935.
[2] C. Biró, P. Hamburger, and A. Pór. Standard examples as subposets of posets. Order, 32:293-299, 2015. arXiv:1311.6518.
[3] C. Biró, P. Hamburger, A. Pór, and W. T. Trotter. Forcing posets with large dimension to contain large standard examples. Graphs and Combinatorics, 32:861-880, 2016. arXiv:1402.5113.
[4] M. Bonamy, L. Esperet, C. Groenland, and A. Scott. Optimal labelling schemes for adjacency, comparability, and reachability. In Proceedings of the 53rd Annual ACM SIGACT Symposium on Theory of Computing, STOC 2021, page 1109-1117. Association for Computing Machinery, 2021. arXiv:2012.01764.
[5] B. Dushnik and E. W. Miller. Partially ordered sets. Amer. J. Math., 63:600-610, 1941.
[6] S. Felsner, T. Mészáros, and P. Micek. Boolean dimension and tree-width. Combinatorica, 40:655-677, 2020. arXiv:1707.06114.
[7] G. Gambosi, J. Nešetřil, and M. Talamo. On locally presented posets. Theoretical Computer Science, $70(2): 251-260,1990$.
[8] J. Holm, E. Rotenberg, and M. Thorup. Planar reachability in linear space and constant time. In 2015 IEEE 56th Annual Symposium on Foundations of Computer Science, pages 370-389, 2015. arXiv:1411.5867.
[9] D. Kelly. On the dimension of partially ordered sets. Discrete Mathematics, 35(1):135-156, 1981. Special Volume on Ordered Sets.
[10] J. Kozik, P. Micek, and W. T. Trotter. Dimension is polynomial in height for posets with planar cover graphs. arXiv:1907.00380, 2019.
[11] T. Mészáros, P. Micek, and W. T. Trotter. Boolean dimension, components and blocks. Order, 2019. arXiv:1801.00288.
[12] J. Nešetřil and P. Pudlák. A note on Boolean dimension of posets. In Irregularities of partitions (Fertőd, 1986), volume 8 of Algorithms Combin. Study Res. Texts, pages 137-140. Springer, Berlin, 1989.
[13] A. Scott and P. Seymour. A survey of $\chi$-boundedness. Journal of Graph Theory, 95(3):473-504, 2020. arXiv:1812.07500.
[14] M. Thorup. Compact oracles for reachability and approximate distances in planar digraphs. Journal of the ACM, 51(6):993-1024, 2004.
[15] W. T. Trotter. Order preserving embeddings of aographs. In Theory and Applications of Graphs, volume 642, pages 572-579. Springer-Verlag, 1978.
[16] W. T. Trotter. Combinatorics and partially ordered sets: Dimension theory. Johns Hopkins Series in the Mathematical Sciences. Johns Hopkins University Press, Baltimore, MD, 1992.
[17] W. T. Trotter, Jr. and J. I. Moore, Jr. The dimension of planar posets. Journal of Combinatorial Theory, Series B, 22(1):54-67, 1977.


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[^1]:    ${ }^{1}$ The original question was posed for posets with planar diagrams. We ask a slightly more general question to establish an immediate connection with reachability labelings.

[^2]:    ${ }^{2}$ Note that this is the only place in the proof we use the assumption that all elements of $\left\{j_{1}, \ldots, j_{k}\right\}$ have the same parity.

