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ORDER PRESERVING EMBEDDINGS OF AOGRAPHS

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Abstract

We call an oriented graph which does not have any directed cycles an aograph. In this paper we discuss the problem of embedding an aograph on a surface in an order preserving fashion. The general problem is motivated by recent research involving partially ordered sets with planar Hasse diagrams.

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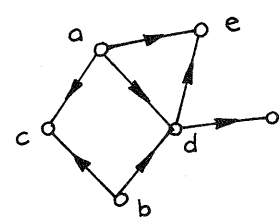

Figure 1a

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We call an oriented graph G which does not have any directed cycles an aograph (short for acyclic oriented graph). With an aograph G, we associate a partial order P(G) on the vertex set of G defined by $a < b$ in P(G) iff there is a directed path from b to a in G. It is convenient to draw a graph diagram of an aograph so that b is higher in the plane than a whenever $a < b$ in P(G). In such a diagram, which we will call an order diagram, it is not necessary to include the orientation on the edges. Figure 1b is an order diagram for the aograph in Figure 1a.

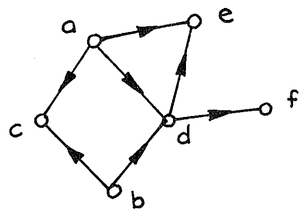

Figure 1a

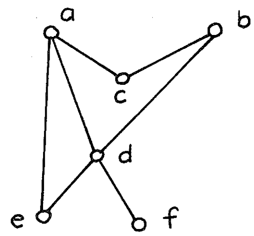

Figure 1b

As is the case with ordinary diagrams for graphs, incidental edge crossings are permitted in order diagrams. We say an aograph is planar when it is possible to draw, in the plane, an order diagram for the aograph with no incidental edge crossings. For example, the aograph in Figure 1 is planar.

Problem 1: Find the minimum list L_1 of aographs so that an aograph is planar if and only if it does not contain a subgraph isomorphic to a graph from L_1.

The aographs in Figure 3 belong to L_1; however, we comment that Problem 1 is most likely a very difficult problem.

A partial order P is said to have an upper bound when there exists a point x so that $y \le x$ for all points y. Lower bounds are defined analogously. A partial order is said to be bounded when it has both an upper bound and a lower bound. We will use the symbol 0 to denote a lower bound and 1 to denote an upper bound. We will say that an aograph G is bounded when P(G) is bounded.

Any Hasse diagram of a partial order is also an order diagram of an aograph. Conversely, a Hasse diagram for P(G) can be obtained from an order diagram of G by removing (if necessary) some of the edges in the diagram.

Dushnik and Miller [1] defined the dimension of a partial order P on a set X, denoted Dim(P), as the smallest positive integer n for which there exist n linear extensions $L_1, L_2, \ldots, L_n$ of P so that $P = L_1 \cap L_2 \cap \ldots \cap L_n$. It is well known that the dimension of a bounded partial order which has a planar Hasse diagram is at most two. Trotter and Moore [3] proved that the dimension of a partial order with a lower bound and a planar Hasse diagram is at most three. Trotter and Moore also gave an infinite family of four dimensional partial orders which have planar Hasse diagrams.

Problem 2: Determine whether planar posets with dimension larger than four exist.

In [3] Trotter and Moore proved that if G is an aograph formed by orienting an ordinary tree (G is also called a tree), then the aograph H obtained from G by adding a point 0 with a directed edge from x to 0 for each $x \in G$ is also planar. We say such an aograph is outerplanar.

Problem 3: Determine the minimum list L_2 of aographs so that an aograph is outerplanar if and only if it does not contain a subgraph isomorphic to a graph from L_2.

Some of the aographs in L_2 are shown in Figure 2.

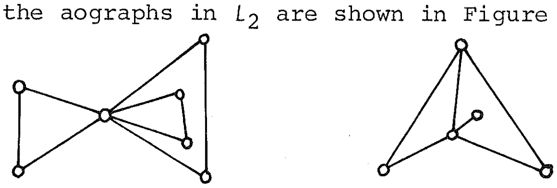

Figure 2

We now discuss the problem of embedding aographs in order preserving fashion on a sphere with n-handles. We consider such spheres in ordinary 3-space using the z-axis to determine downwardness. With the usual notions of piecewise linearity implied but not stated, we define the <u>order preserving genus</u> of an aograph G, denoted $\gamma_d(G)$ to be the smallest positive integer n for which there exists an embedding of G on a sphere with n-handles so that edges do not cross, and whenever there is a directed edge from b to a in G, the edge from b to a in the embedding flows downward.

We note that embedding an aograph on a sphere and on a plane are related but not equivalent problems. For example, the aographs in Figure 3 are non-planar but each has order preserving genus zero.

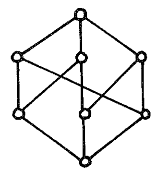

Figure 3a

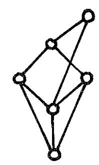

Figure 3b

<u>Problem 4</u>: Find the minimum list L_3 of aographs so that an aograph has order preserving genus zero if and only if it does not contain a subgraph isomorphic to a graph in L_3.

In [2] the author defined for $n \geq 3$ and $k \geq 0$ the crown S_n^k as the poset of height one with maximal elements $\{a_i : 1 \leq i \leq n + k\}$, minimal elements $\{b_i : 1 \leq i \leq n + k\}$, and partial ordering defined cyclically as follows: Each b_i is incomparable with $a_i, a_{i+1}, \ldots, a_{i+k}$ and is less than the remaining $n - 1$ maximal elements. The author proved that the crown S_n^k was a poset of dimension $\{2(n + k)/(k + 2)\}$.

The crown S_n^0 is isomorphic to the 2n-element poset formed by the $n - 1$-element and 1-element subsets of an n-element set ordered by inclusion. In dimension theory, the poset S_n^0 plays an analogous role to the complete graph in chromatic number theory for graphs. e.g.

S_n^0 is the standard example of an n-dimensional poset. The Hasse diagram for S_n^0 has, as its underlying ordinary graph, the complete bipartite graph $K_{n,n}$ minus a 1-factor. It follows that the order preserving genus of S_n^0 is at least as large as the ordinary genus of $K_{n,n}$ - 1-factor. By elementary reasoning, it follows that $\gamma(K_{n,n} - \text{1-factor}) \geq \{(n-1)(n-4)/4\}$. A. T. White and M. Jungerman [6] have made substantial progress towards determining that equality actually holds. It is reasonable to conjecture that $\gamma_d(S_n^0)$ is also $\{(n-1)(n-4)/4\}$.

In view of the results involving the embedding of posets with bounds on the plane, the author conjectured that there existed a function f(n) so that if G is a bounded aograph with $\gamma_d(G) = n$, then $\text{Dim } P(G) \leq f(n)$. It seemed plausible that the techniques of [3] could be modified to produce such a result. However, we will now prove:

<u>Theorem 1</u>: For every $n \geq 3$ and $k \geq 0$, there exists a bounded aograph G with $\gamma_d(G) = 0$ so that the crown S_n^k is a subposet of P(G).

<u>Proof</u>: Given integers $n \geq 3$ and $k \geq 0$, consider an aograph H whose order diagram has a grid-like pattern of the following type.

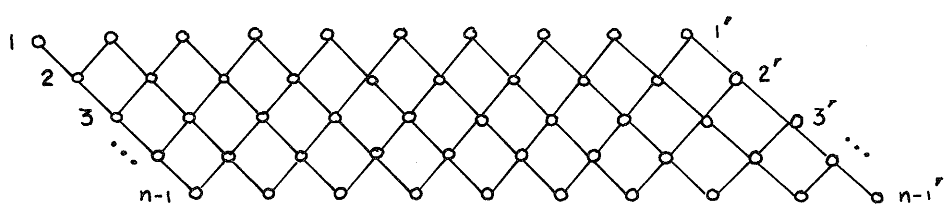

Figure 4

We choose the size of the grid so that P(H) has n + k + 1 maximal elements and the length of the longest chain in P(H) is n - 1.

Now form an aograph G by identifying the points marked i and i' for i = 1, 2,...,n - 1. $\tilde{G}$ is then formed from G by attaching a point

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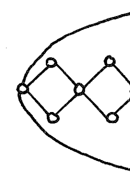

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1 directed to each maximal element of P(G) and a point 0 with an edge from each minimal element of P(G) to 0.

The diagram in Figure 3a is an order diagram for $\tilde{G}$ when n = 3 and k = 0. The subposet of $P(\tilde{G})$ determined by the maximal elements and the minimal elements of P(G) is S_n^k. Finally, we note that $\gamma_d(\tilde{G}) = 0$ since it is easy to embed $\tilde{G}$ on a sphere by wrapping G around the equator with 1 and 0 located at the North and South poles respectively.

As a consequence of this theorem, we see that bounded aographs with order preserving genus zero and arbitrarily large dimension exist. (In fact the aograph $\tilde{G}$ has the same dimension as S_n^k).

It is easy to see that the aograph G is planar only when n is 3 or 4. An embedding of G when n = 4 and k = 5 is shown in Figure 5.

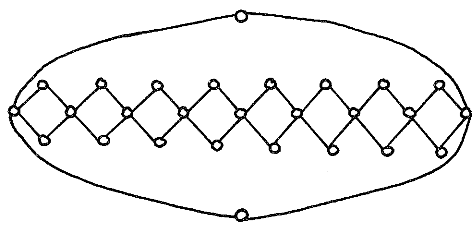

Figure 5

However, the dimension of this aograph is four when k = 0 or 1 and is three when k ≥ 2. In view of the fate of the author's conjecture concerning the dimension of aographs with order preserving genus t, we are reluctant to count this as evidence in support of four as an upper bound on the dimension of planar posets.

In retrospect, the existence of posets with order preserving genus zero and arbitrarily large dimension as established in Theorem 1 is not overly surprising in view of the characterization of dimension in terms of TM-cycles presented in [3]. However, this does suggest another area for investigation, specifically the embedding of aographs in surfaces which do not have an inherent circular nature.

Consider a number of half-planes in 3-space with each half plane containing the points on and to one side of the z-axis. These half-planes form a surface like the pages of a book. It is easy to see that any aograph can be embedded on this surface provided the book has sufficiently many pages; e.g., the aographs in Figure 3 require 3 pages.

Problem 5: For $n \geq 3$, do there exist (bounded) aographs with arbitrarily large dimension which can be embedded in a book with n pages?

In this brief paper we have given some recent results further detailing the interplay between the dimension theory of partial orders and graph theory. We refer the reader to [4] and [5] for other research papers of a similar nature. In the first paper, Trotter and Moore prove that the dimension of the poset consisting of all connected induced subgraphs of a connected graph is the number of non-cut vertices. In the second paper, Trotter, Moore, and Sumner prove that the dimension of a poset depends only on the underlying comparability graph.

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63 (1941) 600-610.

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(1974) 85-103.

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