Linear system of equations

$$A\vec{x} = \vec{b} \qquad A \in R^{n\times p}$$

$$\begin{bmatrix} \vec{A}_1 & \vec{A}_2 & \cdots & \vec{A}_p \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} = \vec{b}$$

Perform elementary row operations.
- Interchange two rows.
- Multiply a row by a non-zero number.
- Add a multiple of one row to another.

Row Echelon Form (REF).

- leading entry / pivot: the first non-zero entry in each row.
- leading zeros: # zeros before the pivot.
- REF: the # leading zeros increases.

Solutions of a linear system:

Look at REF

$p=3$.

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{array}\right] \end{array} \qquad \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$0 = 1$

inconsistent $\Rightarrow$ No solution.

$\text{rank} A = p$.

$\text{rank} A < p$.

Does the linear system has a unique solution /infinite solutions or no solution ?

If the system is consistent

No. $\boxed{0 = *}$

⇓

No solution.

Yes.

rank $A = p$

⇓

a unique solution.

rank $A < p$

⇓

infinite solutions.

Least squares

Linear system. $A\vec{x} = \vec{b}$. $\quad A \in R^{n\times p} \quad A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_p \end{bmatrix}$

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_p \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} = \begin{bmatrix} \vec{b} \end{bmatrix}$$

$$\underbrace{x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_p\vec{v}_p}_{\text{a linear combination of } \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p} = \vec{b}$$

Column space of A / Range of A.

$$V = \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}.$$

$A\vec{x} = \vec{b}$ has solutions $\iff$ $\vec{b} \in V$.

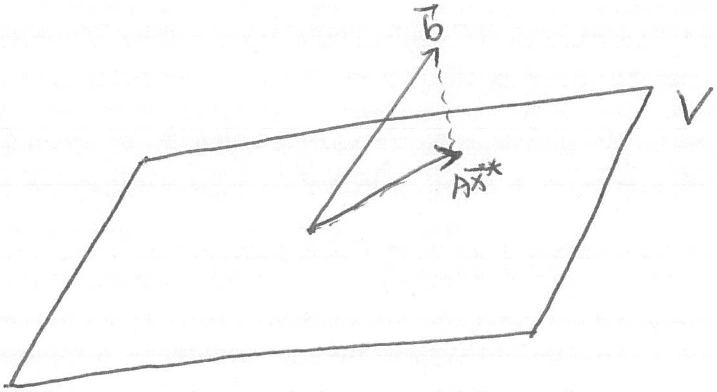
$A\vec{x} = \vec{b}$ does not have solution $\iff$ $\vec{b} \notin V$.

What if $A\vec{x} = \vec{b}$ does not have solutions?

$\Downarrow$

Find the best possible solution

$$\min_{x} \| A\vec{x} - \vec{b} \|^2.$$



<u>Orthogonal projection</u>: the orthogonal projection $\mathrm{Proj}_V \vec{b}$ is the vector in V that is closest to $\vec{b}$.

$$\| \vec{b} - \mathrm{Proj}_V \vec{b} \| \le \| \vec{b} - \vec{v} \| \quad \text{for any } \vec{v} \in V.$$

Instead of solving $A\vec{x} = \vec{b}$, Least Squares suggests solving

$$A\vec{x} = \mathrm{Proj}_V \vec{b}$$

Suppose $\vec{x}^*$ is the least squares solution such that

$$\| \vec{b} - A\vec{x}^* \| \le \| \vec{b} - A\vec{x} \| \quad \text{for all } \vec{x} \in \mathbb{R}^p$$

$$\Updownarrow$$

$$A\vec{x}^* = \mathrm{Proj}_V \vec{b}.$$

$$\Updownarrow$$

$$\vec{b} - A\vec{x}^* \in V^\perp = (\mathrm{Image}(A))^\perp = \ker(A^T)$$

Theorem.
$(\mathrm{Image}\ A)^\perp = \ker(A^T)$

$$\Updownarrow$$

$$A^T(\vec{b} - A\vec{x}^*) = \vec{0}$$

$$\Updownarrow$$

$$A^T A \vec{x}^* = A^T \vec{b}$$

In the case that. $\ker(A) = \{\vec{0}\} \Leftrightarrow \operatorname{rank} A = p$,

$A^T A$ is invertible

The LS solution is given by.

$$\vec{x}^* = (A^T A)^{-1} A^T \vec{b}$$

QR Factorization

Suppose A is an $n \times p$ $(n \geq p)$ matrix with p linearly independent columns. :

$$A = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_p \\ | & & | \end{bmatrix}$$

where $\vec{v}_1, \vec{v}_2 \cdots \vec{v}_p$ are linearly independent.

$\Leftrightarrow \operatorname{rank} A = p$.

Then there exists an $n \times p$ matrix $Q = \begin{bmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_p \\ | & & | \end{bmatrix}$

whose columns $\vec{u}_1, \cdots, \vec{u}_p$ are orthonormal,

and an upper triangular matrix R with positive diagonal entries such that.

$$A = QR.$$

$$A \quad = \quad Q \quad R$$

$$\begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_p \end{bmatrix} = \begin{bmatrix} \vec{u}_1 & \cdots & \vec{u}_p \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1p} \\ 0 & R_{22} & & R_{2p} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & R_{pp} \end{bmatrix}$$

Proof: By Gram-Schmidt Process.

LS and QR Factorization

$$A = QR.$$

$$A^TA = R^TQ^TQR = R^TR \qquad (A^TA)^{-1} = R^{-1}(R^T)^{-1}$$

$$\begin{aligned} \vec{x}^* &= (A^TA)^{-1} A^T\vec{b} \\ &= R^{-1}(R^T)^{-1} R^T Q^T \vec{b} \\ &= R^{-1} Q^T \vec{b} \end{aligned}$$

Solving $R\vec{x} = Q^T\vec{b}$ is easy since R is upper-triangular

Colinearity.

$$A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_p \end{bmatrix}$$

Colinearity refers to the situation where $\vec{v}_1\ \vec{v}_2 \cdots \vec{v}_p$ are linearly dependent : $\text{rank}(A) < p$

In this case. A^TA is not invertible.

Example:

$$A = \begin{array}{c} \begin{array}{ccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{array} \\ \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & -1 \\ 1 & 2 & -1 \\ 1 & 2 & 1 \end{bmatrix} \end{array} \qquad \vec{v}_2 = 2\vec{v}_1$$

- $\text{rank}(A^TA) = 2$, A^TA is not invertible.
- The LS solution $\min_x \| A\vec{x} - \vec{b} \|$ is not unique.

Why: If $\begin{pmatrix} x_1^* \\ x_2^* \\ x_3^* \end{pmatrix}$ is a solution, there are other solutions like.

$$\begin{pmatrix} x_1^* + 2x_2^* \\ 0 \\ x_3^* \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 \\ \frac{1}{2}x_1^* + x_2^* \\ x_3^* \end{pmatrix}$$