Hermite Interpolation: the interpolation of a function and some of its derivatives at a set of nodes.

Example 1: \[ P(x_0) = f(x_0) \quad P(x_1) = f(x_1) \]
\[ P'(x_0) = f'(x_0) \quad P'(x_1) = f'(x_1). \]

Write \[ P(x) = a + b(x - x_0) + c(x - x_0)^2 + d(x - x_0)^2(x - x_1), \]
\[ P'(x) = b + 2c(x - x_0) + 2d(x - x_0)(x - x_1) + d(x - x_0)^2. \]

The four conditions become:
\[ \begin{align*}
  f(x_0) &= a \\
  f'(x_0) &= b \\
  f(x_1) &= a + bh + ch^2, \quad h = x_1 - x_0 \\
  f'(x_1) &= b + 2ch + dh^2.
\end{align*} \]

There exists a unique solution.

Example 2. \[ P(0) = 0 \quad P(1) = 1 \quad P'\left(\frac{1}{2}\right) = 2. \]

Write \[ P(x) = a + bx + cx^2. \]
\[ P(0) = a = 0, \]
\[ P(1) = b + c = 1 \]
\[ P'\left(\frac{1}{2}\right) = b + c = 2 \]
\[ \Rightarrow \text{no solution.} \]

One can interpolate with a higher order polynomial.
\[ P(x) = a + bx + cx^2 + dx^3. \]
\[ P(0) = a = 0, \]
\[ P(1) = b + c + d = 1 \]
\[ P'\left(\frac{1}{2}\right) = b + c + \frac{3d}{4} = 2. \]
\[ \Rightarrow \frac{d}{4} = -1 \quad d = -4 \quad b + c = 5. \]
Hermite interpolation: whenever a derivative $p^{(j)}(x_i)$ is to be prescribed, $p^{(j-1)}(x_i), \ldots, p'(x_i)$ and $p(x_i)$ will also be prescribed.

Nodes: $x_0, x_1, \ldots, x_n$ \{ $x_i$ \}_{i=0}^{n}

Hermite problem: $p^{(j)}(x_i) = C_{ij}$ \hspace{1em} $0 \leq j \leq k_i - 1$ \hspace{1em} $0 \leq i \leq n$.

Degree of freedom: $m+1 = k_0 + k_1 + \cdots + k_n$.

Interpolate with an $m$th order polynomial.

Uniqueness.

There exists a unique polynomial of degree $\leq m$ which satisfies the Hermite interpolation conditions above.

Proof: • If there are two polynomials $p(x)$, $\tilde{p}(x)$, then $p(x) - \tilde{p}(x)$ is of degree $\leq m$, and interpolate zero function.

• So we only need to prove the homogeneous problem is unique.

\[ p^{(j)}(x_i) = 0, \hspace{1em} 0 \leq j \leq k_i - 1 \hspace{1em} 0 \leq i \leq n. \Rightarrow p = 0. \]

• Such $p(x)$ has a zero of multiplicity $k_i$ at $x_i$ ($0 \leq i \leq n$) must be like

\[ p(x) = C \frac{\prod_{i=0}^{n} (x-x_i)^{k_i}}{k_0 + k_1 + \cdots + k_n = m} \]

However, $p(x)$ is of degree $\leq m$. Contradiction: $p(x) = 0$.

Example 3. Interpolation at a single node.

$p^j(x_0) = C_{0j}$ \hspace{1em} $0 \leq j \leq k.$

\[ p(x) = C_{00} + C_{01} (x-x_0) + \frac{C_{02}}{2!} (x-x_0)^2 + \cdots + \frac{C_{0k}}{k!} (x-x_0)^k \]
Example 4. \( p(x_0) = C_0 \quad p'(x_0) = C_1 \quad p(x_1) = C_{10} \)

Divided difference Table:

| \( x_0 \) | \( C_0 \) | \( C_{0,1} = \frac{p[x_0, x_0]}{x_0 - x_0} \) |
| \( x_0 \) | \( C_0 \) | \( C_{10} = \frac{p[x_0, x_0]}{x_1 - x_0} \) |
| \( x_0 \) | \( C_0 \) | \( C_{10} = \frac{p[x_0, x_0]}{x_1 - x_0} \) |

How to define \( f[x_0, x_0] \):

\[
\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \quad \Rightarrow \quad f[x_0, x_0] = f'(x_0)
\]

\[ p(x) = p(x_0) + p[x_0, x_0] (x - x_0) + p[x_0, x_0, x_1] (x - x_0)^2. \]

Example 5. \( p(x_0) = f(x_0) \quad p'(x_0) = f'(x_0) \quad p(x_1) = f(x_1) \quad p'(x_1) = f'(x_1) \)

Table:

| \( x_0 \) | \( f(x_0) \) | \( p[x_0, x_0] = f'(x_0) \) |
| \( x_0 \) | \( f(x_0) \) | \( p[x_0, x_1] \) |
| \( x_1 \) | \( f(x_1) \) | \( p[x_1, x_1] = f'(x_1) \) |

\[ p(x) = f(x_0)(x - x_0) - f'(x_0)(x - x_0) + p[x_0, x_0, x_1] (x - x_0)^2. \]

Divided difference and derivatives:

If \( x_0, x_1, \ldots, x_k \in [a, b] \) are the smallest interval containing \( x_0, \ldots, x_k \), then there exists \( s \in [a, b] \) such that

\[
f[x_0, x_1, \ldots, x_k] = \frac{1}{k!} f^{(k)}(s)
\]

Let this interval shrink to 0, \( f[x_0, x_0, \ldots, x_0] = \frac{1}{k!} f^{(k)}(x_0) \).
Example 6. \( p(1) = 2 \), \( p'(1) = 3 \), \( p(2) = 6 \), \( p'(2) = 7 \), \( p''(2) = 8 \).

\[
\begin{align*}
1 & \quad 2 & \quad 3 = p(1) & \quad p(1,1) = 1 & \quad p(1,1,2) = 1 & \quad p(1,1,2,3) = -1. \\
1 & \quad 2 & \quad p(1,2) = 4 & \quad p(1,2,1) = \frac{T-4}{1} = 3 & \quad p(1,2,1,2) = 1 \\
2 & \quad 6 & \quad p(2,1) = 2 & \quad p(2,1,2) = \frac{p''(2)}{2} = 4. \\
2 & \quad 6 & \quad p(2,2) \\
\end{align*}
\]

\[
p(x) = 2 + 3(x-x_1) + (x-1) + 2(x-1)^2(x-2) + (-1)(x-1)^2(x-2)^2.
\]

**Lagrange Form**

\[
P(x_i) = C_{i0} \quad P'(x_i) = C_{i1} \quad 0 \leq i \leq n.
\]

**Lagrange polynomial:**

\[
P(x) = \sum_{i=0}^{n} C_{i0} A_i(x) + \sum_{i=0}^{n} C_{i1} B_i(x)
\]

\[
\begin{cases}
A_i(x_j) = \delta_{ij} & A_i(x_j)' = 0 \\
B_i(x_j) = 0 & B_i(x_j)' = \delta_{ij}
\end{cases}
\]

**How to construct such \( A_i \)'s and \( B_i \)'s.**

\[
L_i(x) = \frac{\prod_{j=0}^{n} x-x_j}{\prod_{j \neq i} x_i-x_j}
\]

Define \[ A_i(x) = \left[ 1 - 2(x-x_i) \cdot L_i'(x_i) \right] L_i(x) \]

\[ B_i(x) = (x-x_i) L_i^2(x) \]

**Error estimate on Hermite interpolation**

Let \( x_0, x_1, \ldots, x_n \) be distinct nodes in \([a,b]\) and \( f \in C^{2n+2} [a,b]\).

If \( p \) is the polynomial of degree \( 2n+1 \) such that

\[
p(x_i) = f(x_i) \quad p'(x_i) = f'(x_i) \quad i=0,1, \ldots, n.
\]

Then to each \( x \in [a,b] \) there exists \( \xi \in (a,b) \) such that

\[
f(x) - p(x) = \frac{\int_{a}^{b} (2n+2) \, t^{(2n+2)} \, (x-t)^2 \, dt}{(2n+2)!} \sum_{i=0}^{n} \frac{1}{i!} (x-x_i)^2.
\]
Proof: \[ \phi = f - p - \lambda x w \quad \text{with} \quad \lambda = \frac{n}{\sum_{i=0}^{n} (t-x_i)^2} \]

Choose \( \lambda x \) such that \( \phi(x) = 0 \) by letting \( \lambda x = \frac{f(x) - p(x)}{w(x)} \).

\( \phi \) has at least \( n+2 \) zeros, namely, \( x, x_0, x_1, \ldots, x_n \).

By Rolle's theorem, \( \phi' \) has at least \( n+1 \) zeros different from \( x, x_0 \).

Also \( \phi'(x_0) = 0, \phi'(x_1) = 0 \ldots \phi'(x_n) = 0 \).

Then \( \phi' \) has at least \( 2(n+1) \) zeros.

\( \phi^{(2n+2)} \) has one zero, denoted by \( \xi \).

\[ 0 = \phi^{(2n+2)}(\xi) = f^{(2n+2)}(\xi) - p^{(2n+2)}(\xi) - \lambda x^{(2n+2)}(\xi) \]

\[ \Rightarrow \lambda x = \frac{f^{(2n+2)}(\xi)}{2n+2)!} \]

Interpolation with repeated nodes.

\( p \) interpolates \( f \) at \( 1, 3, 8, 1, 13, 1, 8 \).

\[ p(1) = f(1) \quad p(3) = f(3) \quad p(8) = f(8) \quad p(13) = f(13) \]

\[ p'(1) = f'(1) \quad p'(8) = f'(8) \]

\[ p''(1) = f''(1) \]

Uniqueness.

Let \( x_0, x_1, \ldots, x_m \) be a list of points in which no element is repeated more than \( k \) times. Let \( f \in C^{k+1} \) on an interval containing these points. Then there exists a unique polynomial of degree \( \leq m \) that interpolates \( f \) at the given points.

General Newton Interpolating Formula

\[ p(x) = \prod_{j=0}^{m} f[x_0, x_1, \ldots, x_j] \prod_{i=0}^{j-1} (x-x_i) \]
General Newton Interpolation Polynomial.

\[ p(x) = \sum_{j=0}^{n} f[x_0, x_1, \ldots, x_j] \prod_{i=0}^{j-1} (x - x_i). \]

Thus, if \( f \) is sufficiently differentiable so that the divided differences in the equation above exist, then \( p(x) \) gives the polynomial of degree \( \leq n \) that interpolates \( f \) at \( x_0, \ldots, x_n \).

Recursive formula.

Let \( x_0 \leq x_1 \leq \ldots \leq x_n \).

\[ f[x_0, x_1, \ldots, x_n] = \begin{cases} \frac{f[x_1, \ldots, x_n] - f[x_0, \ldots, x_{n-1}]}{x_n - x_0} & \text{if } x_n \neq x_0 \\ \frac{f^{(n)}(x_0)}{n!} & \text{if } x_n = x_0. \end{cases} \]

Proof by induction.

\( \text{n=1. } p_1(x) = \begin{cases} (x-x_0) \frac{f(x_1) - f(x_0)}{x_1 - x_0} + f(x_0) & x_1 \neq x_0. \\ f'(x_0)(x_0 - x_0) + f(x_0) & x_0 = x_1. \end{cases} \)

\[ f[x_0, x_1] = \begin{cases} \frac{f(x_1) - f(x_0)}{x_1 - x_0} & \text{if } x_1 \neq x_0. \\ f'(x_0) & \text{if } x_1 = x_0. \end{cases} \]

\( \text{n-1 } P_{n-1}(x) \text{ interpolates } f \text{ at } x_0, \ldots, x_{n-1} \)

\( \text{n. } \text{if } x_n \neq x_0 \text{ proof is the same as before.} \)

\( \text{if } x_n = x_0 \) \[ P_n(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_0) (x - x_0)^k. \]

The coefficient of \( x^n \) is \( \frac{f^{(n)}(x_0)}{n!} \).