Midterm 1 – 4640, Spring 2020
Instructor: Wenjing Liao

• Test time: 75 minutes.
• Please do not assist another person in the completion of this exam. Please do not copy answers from another student’s exam. Please do not have another student take your exam for you. Please keep your own work covered up as much as possible during the exam so that others will not be tempted or distracted.
• No notes and books are allowed.
• You are encouraged to use calculators and keep six digits.
• Read each problem carefully. **Show all work for full credit.**
• Make sure you have **11 pages**, including the cover page (Page 1), the page of scores (Page 2), and two blank pages (Page 10 and 11).

Your name: ________________
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Problem 1: (16 points): You may use the following geometric sum
\[ 1 + q + q^2 + \ldots + q^{n-1} = \frac{1 - q^n}{1 - q}, \text{ for } |q| < 1. \]

(1) Convert the following number to its decimal equivalent
\[ (.ABABAB\ldots)_{16}. \]
(5 points)

(2) Let \( x = (.ABABAB\ldots)_{16} \) and its approximation be \( x_A = (.AB)_{16} \). What is the relative error in this approximation? (5 points)
(3) For the following numbers $x_A$ and $x_T$, how many significant digits are there in $x_A$ with respect to $x_T$?

- $x_T = 0.00819256, x_A = 0.00819$; (3 points)
- $x_T = 453.1415926, x_A = 453.1415$. (3 points)
Problem 2 (20 points): This problem is to find the intersection of $y = x$ and $y = \cos x$ when $x \in (0, \frac{\pi}{2})$. It is equivalent to find the root of

$$f(x) = \cos x - x.$$ 

(1) Use Newton’s method to find the intersection with the starting point $x_0 = \frac{\pi}{4}$.

- Write down Newton’s method about the update from $x_n$ to $x_{n+1}$. (5 points)

- Use calculators to compute $x_1$ and $x_2$. (4 points)
(2) Suppose $\alpha$ is the root of $f$ such that $f(\alpha) = 0$. Write down the definition that a sequence $\{x_n | n \geq 0\}$ converges to $\alpha$ with order $p \geq 1$. (4 points)

(3) What is the order of convergence for the Newton’s method? Assume $\max_x |f''(x)| \leq M$ and $\min_x |f'(x)| \geq m$ for some $M > 0$ and $m > 0$. Explain. (7 points)
Problem 3 (12 points): Use Newton divided difference method to compute a polynomial $p(x)$ of degree at most 6 such that $p(0) = 1, p'(0) = 0, p''(0) = 2, p'''(0) = -6$ and $p(1) = 2, p'(1) = 0, p(2) = 1$. (Keep the final answer in the Newton’s form and DO NOT simplify.)
Problem 4 (12 points): Suppose we interpolate the function 

\[ f(x) = \sin(\pi x) \]

with the polynomial \( p_n \) of degree at most \( n \) using \( n + 1 \) Chebychev nodes \( x_0, \ldots, x_n \) in \([-1, 1]\).

(1) What is a good upper bound for \( \| f - p_n \|_\infty \)? Recall that \( \| f - p_n \|_\infty := \max_{x \in [-1,1]} |f(x) - p_n(x)| \). (8 points)

(2) How many Chebyshev nodes should we use to guarantee \( \| f - p_n \|_\infty < 0.01 \)? (4 points)
Problem 5 (10 points): Suppose real numbers are stored in the following binary form
\[ x = \sigma \cdot (a_1a_2\ldots a_t a_{t+1}\ldots)_2 \cdot 2^e, \ a_1 \neq 0, \]
where \( \sigma = 1 \) or \( -1 \). The chopped machine representation of \( x \) is given by
\[ \text{fl}(x) = \sigma \cdot (a_1a_2\ldots a_{16})_2 \cdot 2^e, \ a_1 \neq 0. \]
Which of the followings is true?
\[ \text{fl}(1 + 2^{-16}) = 1 \text{ or } \text{fl}(1 + 2^{-16}) > 1? \]
Explain.