Problem 1: (16 points): You may use the following geometric sum
\[ 1 + q + q^2 + \ldots + q^{n-1} = \frac{1 - q^n}{1 - q}, \text{ for } |q| < 1. \]

(1) Convert the following number to its decimal equivalent
\[(ABABAB\ldots)_{16}.\]

(5 points)
\[
\begin{align*}
&= 10 \cdot 16^{-4} + 11 \cdot 16^{-3} + 10 \cdot 16^{-2} + 11 \cdot 16^{-1} + 10 \cdot 16^{-5} + 11 \cdot 16^{-6} + \\
&= 10 \cdot 16^{-1} \left( 1 + 16^{-2} + 16^{-4} + \ldots \right) \\
&+ 11 \cdot 16^{-2} \left( 1 + 16^{-2} + 16^{-4} + \ldots \right) \\
&= \left( \frac{10}{16} + \frac{11}{256} \right) \frac{1}{1 - 16^{-2}} = \frac{171}{256} \cdot \frac{1}{1 - \frac{1}{16}} = \frac{171}{255} \\
&= .016706
\end{align*}
\]

(2) Let \( x = (ABABAB\ldots)_{16} \) and its approximation be \( x_A = (AB)_{16}. \) What is the relative error in this approximation?

(5 points)
\[
\begin{align*}
&x = 0.16706 \\
&x_A = 10 \cdot 16^{-1} + 11 \cdot 16^{-2} = \frac{171}{256} \\
\text{relative error} &= \frac{|x - x_A|}{x} = \frac{\frac{171}{256} - \frac{171}{255}}{\frac{171}{255}} \\
&= 1 - \frac{255}{256} = \frac{1}{256} \\
&= 0.0039
\end{align*}
\]
(3) For the following numbers $x_T$ and $x_A$, how many significant digits are there in $x_A$ with respect to $x_T$?

- $x_T = 0.0081256$, $x_A = 0.00819$; (3 points)

\[ x_T - x_A = 0.00000696 \]

3 significant digits.

- $x_T = 483.1415926$, $x_A = 483.1415$. (3 points)

\[ x_T - x_A = 0.0000026 \]

6 significant digits.
Problem 2 (20 points): This problem is to find the intersection of $y = x$ and $y = \cos x$ when $x \in (0, \frac{\pi}{2})$. It is equivalent to finding the root of

$$f(x) = \cos x - x.$$ 

(1) Use Newton's method to find the intersection with the starting point $x_0 = \frac{\pi}{4}$.

- Write down Newton's method about the update from $x_n$ to $x_{n+1}$. (5 points)

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}
\]

\[
x_{n+1} = x_n - \frac{\cos x_n - x_n}{-\sin x_n - 1}
\]

\[
x_{n+1} = x_n + \frac{\cos x_n - x_n}{1 + \sin x_n}
\]

- Use calculators to compute $x_1$ and $x_2$. (4 points)

\[
x_0 = \frac{\pi}{4}
\]

\[
x_1 = \frac{\pi}{4} + \frac{\cos \frac{\pi}{4} - \frac{\pi}{4}}{1 + \sin \frac{\pi}{4}}
\]

\[
x_1 = 0.7375
\]

\[
x_2 = x_1 + \frac{\cos x_1 - x_1}{1 + \sin x_1}
\]

\[
x_2 = 0.7391
\]
(2) Suppose \( \alpha \) is the root of \( f \) such that \( f(\alpha) = 0 \). Write down the definition that a sequence \( \{x_n \mid n \geq 0\} \) converges to \( \alpha \) with order \( p \geq 1 \). (4 points)

\[
|x_{n+1} - \alpha| \leq C |x_n - \alpha|^p
\]

for some constant \( C > 0 \).

(3) What is the order of convergence for the Newton's method? Assume \( \max_x |f'(x)| \leq M \) and \( \min_x |f'(x)| \geq m \) for some \( M > 0 \) and \( m > 0 \). Explain. (7 points)

Consider Taylor expansion of \( f \) at \( (x_n, f(x_n)) \):

\[
f(x) = f(x_n) + (x - x_n) f'(x_n) + \frac{(x - x_n)^2}{2} f''(\xi) \quad \xi \text{ between } x \text{ and } x_n
\]

Let \( x = \alpha \):

\[
f(x) = f(x_n) + (\alpha - x_n) f'(x_n) + \frac{(\alpha - x_n)^2}{2} f''(\xi) \quad \xi \text{ between } \alpha \text{ and } x_n
\]

Let \( f'(x) = 0 \):

\[
0 = \left( f(x_n) \right)' + (\alpha - x_n) f'(x_n) + \frac{(\alpha - x_n)^2}{2} f''(\xi)
\]

so

\[
x_{n+1} - \alpha = (\alpha - x_n)^2 \frac{f''(\xi)}{2 f'(x_n)}
\]

Since \( \left| \frac{f''(\xi)}{2 f'(x_n)} \right| \leq \frac{M}{2m} \),

\[
|x_{n+1} - \alpha| \leq \left( \frac{M}{2m} \right)^p |x_n - \alpha|^p
\]

Second-order convergence, \( p = 2 \).
Problem 3 (12 points): Use Newton divided difference method to compute a polynomial $p(x)$ of degree at most 6 such that $p(0) = 1, p'(0) = 0, p''(0) = -6$ and $p(1) = 2, p'(1) = 0, p(2) = 1$. (Keep the final answer in the Newton’s form and DO NOT simplify.)

\[
\begin{array}{cccccccc}
\hline
x_i & & c_0 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\
\hline
0 & 1 & p(0) = 1 & p(0) = 1 & p(0) = 1 & p(0) = 1 & p(0) = 1 & p(0) = 1 & p(0) = 1 \\
0 & 1 & p(0,0) = 0 & p(0,0) = 0 & p(0,0) = 0 & p(0,0) = 0 & p(0,0) = 0 & p(0,0) = 0 & p(0,0) = 0 \\
0 & 1 & p(0,1) = 0 & p(0,1) = 0 & p(0,1) = 0 & p(0,1) = 0 & p(0,1) = 0 & p(0,1) = 0 & p(0,1) = 0 \\
0 & 1 & p(0,2) = 0 & p(0,2) = 0 & p(0,2) = 0 & p(0,2) = 0 & p(0,2) = 0 & p(0,2) = 0 & p(0,2) = 0 \\
1 & 2 & p(1) = 1 & p(1) = 1 & p(1) = 1 & p(1) = 1 & p(1) = 1 & p(1) = 1 & p(1) = 1 \\
1 & 2 & p(1,1) = 0 & p(1,1) = 0 & p(1,1) = 0 & p(1,1) = 0 & p(1,1) = 0 & p(1,1) = 0 & p(1,1) = 0 \\
1 & 2 & p(1,2) = 0 & p(1,2) = 0 & p(1,2) = 0 & p(1,2) = 0 & p(1,2) = 0 & p(1,2) = 0 & p(1,2) = 0 \\
\hline
\end{array}
\]

2. 1

\[p(x) = c_0 + c_1(x-0) + c_2(x-0)^2 + c_3(x-0)^3 + c_4(x-0)^4 + c_5(x-0)^5(x-1) + c_6(x-0)^6(x-1)^2\]

\[= 1 + (x-0) + (-1)(x-0)^2 + (x-0)^3 + (-3)(x-0)^4(x-1) + \frac{9}{4}(x-0)^5(x-1)^2\]
Problem 4 (12 points): Suppose we interpolate the function
\[ f(x) = \sin(x^2) \]
with the polynomial \( p_n \) of degree at most \( n \) using \( n + 1 \) Chebyshew nodes \( x_0, \ldots, x_n \) in \([-1, 1]\).

(1) What is a good upper bound for \( \| f - p_n \|_\infty \)? Recall that
\[ \| f - p_n \|_\infty := \max_{x \in [-1, 1]} |f(x) - p_n(x)|. \] (8 points)

By the theorem of polynomial approximation error,
\[ \| f - p_n \|_\infty \leq \max_{0 \leq m \leq n} \left| \frac{f^{(m)}(5)}{(m+1)!} \right| \max_{x \in [-1, 1]} \left| \frac{5}{2^{m+1}} (x - x_i) \right|. \]

\[ \left| f^{(m)}(5) \right| \leq m! \text{ and } \max_{x \in [-1, 1]} \left| \frac{5}{2^{m+1}} (x - x_i) \right| = 2^{-m} \text{ with Chebyshev nodes}. \]

\[ \| f - p_n \|_\infty \leq \frac{5^{n+1}}{2^n (n+1)!}. \]

(2) How many Chebyshev nodes should we use to guarantee
\[ \| f - p_n \|_\infty < 0.01? \] (4 points)

\[ n = 5 \quad \text{upper bound} = 0.0417 \]
\[ n = 6 \quad \text{upper bound} = 0.0094 \]

We need at least \( n = 6 \).
\[ \Rightarrow 7 \text{ Chebyshev nodes}. \]
Problem 5 (10 points): Suppose real numbers are stored in the following binary form
\[ x = \sigma \cdot (a_0a_1 \ldots a_{i-1} \ldots 2^i, a_i \neq 0, \]
where \( \sigma = 1 \) or \(-1\). The chopped machine representation of \( x \) is given by
\[ \hat{x}(x) = \sigma \cdot (a_0a_1 \ldots a_{i-1}) \cdot 2^i, a_i \neq 0. \]
Which of the followings is true?
\[ f(1 + 2^{-16}) = 1 \text{ or } \hat{f}(1 + 2^{-16}) > 1? \]
Explain.

\[ 1 = ( .1 \ 0 \ 0 \ \ldots \ 0 ) \cdot 2^0 \]
\[ 2^{-16} = ( .1 \ 0 \ 0 \ \ldots \ 0 ) \cdot 2^{-16} \]
\[ 1 + 2^{-16} = \left[ ( .1 \ 0 \ \ldots \ ) + ( .0 \ 0 \ \ldots \ 0 \ 1 ) \right] 2^0 \]
\[ = \left[ .1 \ 0 \ \ldots \ \uparrow 17^{th} \ \text{position} \right] 2^0 \]
\[ \text{Chopping} \left( .1 \ 0 \ \ldots \ \uparrow 16^{th} \right) 2^0 = 1 \]
\[ f(1 + 2^{-16}) = 1 \]