Test time: 48 hours from Wednesday noon to Friday noon. You can spend any time on this exam within these 48 hours.

This is an open-book take home exam. You can refer to books, notes, calculators and online resources.

You can not communicate with anybody for the completion of this exam. Please do not assist another person in the completion of this exam. Please do not copy answers from another student’s exam. Please do not have other people take your exam for you.

Read each problem carefully. **Show all work for full credit.**

Keep six digits in numerical problems.

Please write solutions for each problem on separate pages. In other words, start a new page when you start a new problem.

Your name: ______________
Problem T1 (6 points): A natural cubic spline $S$ is defined by

$$S(x) = \begin{cases} 
S_0(x) = 1 + B(x - 1) - D(x - 1)^3 & x \in [1, 2) \\
S_1(x) = 1 + b(x - 2) - \frac{3}{4}(x - 2)^2 + d(x - 2)^3 & x \in [2, 3].
\end{cases}$$

If $S$ interpolates the data $(1, 1), (2, 1)$ and $(3, 0)$, find $B, D, b, d$.

\begin{align*}
S_0(x) &= 1 + B(x - 1) - D(x - 1)^3 \\
S_1(x) &= 1 + b(x - 2) - \frac{3}{4}(x - 2)^2 + d(x - 2)^3 \\
S_0'(x) &= B - 3D(x - 1)^2 \\
S_1'(x) &= b - \frac{3}{2}(x - 2) + 3d(x - 2)^2 \\
S_0''(x) &= -6D(x - 1) \\
S_1''(x) &= -\frac{3}{2} + 6d(x - 2)
\end{align*}

Set $S_0(1) = 1 \quad S_0(2) = S_1(2) = 1 \quad S_1(3) = 0$

\begin{align*}
S_0(2) &= 1 \quad 1 + B - D = 1 \quad \Rightarrow \quad B = D \\
S_1(3) &= 0 \quad 1 + b - \frac{3}{2} + d = 0 \quad \Rightarrow \quad b + d = -\frac{3}{2}
\end{align*}

Set $S_0'(2) = S_1'(2) \quad B - 3D = b$

\begin{align*}
S_0''(2) &= S_1''(2) \quad -6D = -\frac{3}{2} \quad \Rightarrow \quad D = \frac{1}{4} \\
\text{Then} \quad B &= \frac{1}{4}
\end{align*}

Consider the natural cubic spline

\begin{align*}
S_0''(1) &= 0 \quad \checkmark \\
S_1''(3) &= -\frac{3}{2} + bd = 0 \quad \Rightarrow \quad d = \frac{1}{4}
\end{align*}

Then $b = -d - \frac{1}{4} = -\frac{1}{2}$.

Answer $B = D = \frac{1}{4} \quad d = \frac{1}{4} \quad b = -\frac{1}{2}$.
Problem T2 (10 points):

(1) What type of bivariate polynomial would be suitable for interpolation on the sets of notes given below. Explain. (3 points)

\[ P(x, y) = \frac{3}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} f(x_i, y_j) V_j(y) U_i(x) \]

where

\[ U_i(x) = \frac{\prod_{k=1, k \neq i}^{3} (x - x_k)}{(x_i - x_k)} \]
\[ V_j(y) = \frac{\prod_{k=1, k \neq j}^{3} (y - y_k)}{(y_j - y_k)} \]

(2) What type of bivariate polynomial would be suitable for interpolation on the sets of notes given below. Explain. (3 points)

Three lines \( L_0 \) has one node
\( L_1 \) has two nodes
\( L_3 \) has three nodes

There is a polynomial of degree \( \leq 2 \) for such interpolation,

\[ a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 y^2 + a_5 x y \]
Consider Shepard interpolation in two variables with 5 nodes and let the function \( \Phi \) be \( \Phi(\vec{p}, \vec{q}) = ||\vec{p} - \vec{q}||_{L_1} = |p_1 - q_1| + |p_2 - q_2| \) where \( \vec{p} = (p_1, p_2) \) and \( \vec{q} = (q_1, q_2) \). Write down the Shepard interpolation function with the following nodes and function values. Do not simply your answer. (4 points)

<table>
<thead>
<tr>
<th>Nodes</th>
<th>(0,0)</th>
<th>(1,1)</th>
<th>(2,-1)</th>
<th>(1,-1)</th>
<th>(-1,-1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Function values</td>
<td>5</td>
<td>-2</td>
<td>7</td>
<td>-3</td>
<td>-1</td>
</tr>
</tbody>
</table>

Shepard interpolant \( F = \sum_{i=1}^{5} f(\vec{p}_i) u_i \)

where \( u_i(\vec{p}) = \prod \frac{\Phi(\vec{p}, \vec{p}_j)}{\Phi(\vec{p}_i, \vec{p}_j)} \)

\[
F(x, y) = 5 \cdot \frac{|x-1|+|y-1|}{|0-1|+|0-1|} \cdot \frac{|x-2|+|y+1|}{|1-2|+|0+1|} \cdot \frac{|x-1|+|y+1|}{|0-1|+|0+1|} \cdot \frac{|x+1|+|y+1|}{|1+1|+|0+1|} \\
-2 \cdot \frac{|x-0|+|y-0|}{1+1} \cdot \frac{|x-2|+|y+1|}{|1-2|+|1+1|} \cdot \frac{|x-1|+|y+1|}{|1-1|+|1+1|} \cdot \frac{|x+1|+|y+1|}{2+2} \\
+7 \cdot \frac{|x+1|+|y|}{2+1} \cdot \frac{|x-1|+|y-1|}{|2+1|+|-1-1|} \cdot \frac{|x-1|+|y+1|}{|2-1|+|1+1|} \cdot \frac{|x+1|+|y+1|}{|2+1|+|0|} \\
-3 \cdot \frac{|x+1|+|y|}{1+1} \cdot \frac{|x-1|+|y-1|}{|1-1|+|1-1|} \cdot \frac{|x-2|+|y+1|}{|1-2|+|1+1|} \cdot \frac{|x+1|+|y+1|}{|1+1|+|1+1|} \\
- \frac{|x+1|+|y|}{1+1} \cdot \frac{|x-1|+|y-1|}{|1-1|+|1-1|} \cdot \frac{|x-2|+|y+1|}{|1-2|+|1+1|} \cdot \frac{|x-1|+|y+1|}{|-1-1|+|-1-1|} \\
\]
Problem T3 (8 points): Calculate the polynomial \( p(x) \) of degree \( \leq 3 \) which minimizes
\[
\int_{-1}^{1} (e^{|x|} - p(x))^2 \, dx.
\]

This problem is equivalent to find \( p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \) to best approximate \( f(x) = e^{|x|} \)
with respect to the inner product \( <f, g> = \int_{-1}^{1} f(x) g(x) \, dx \)

According to HW, \( f(x) \) is an even function, \( p(x) \) should be even: \( a_1 = 0 \), \( a_3 = 0 \)

\[
p(x) = a_0 + a_2 x^2 \quad g_0(x) = 1 \quad g_2(x) = x^2
\]

\[
= a_0 g_0 + a_2 g_2
\]

\[
\begin{bmatrix}
  <g_0, g_0> & <g_0, g_2> \\
  <g_2, g_0> & <g_2, g_2>
\end{bmatrix}
\begin{bmatrix}
  a_0 \\
  a_2
\end{bmatrix}
= \begin{bmatrix}
  <f, g_0> \\
  <f, g_2>
\end{bmatrix}
\]

\[
<g_0, g_0> = \int_{-1}^{1} 1 \, dx = 2 \quad <g_0, g_2> = \int_{-1}^{1} x^2 \, dx = \frac{4}{3} x^3 \bigg|_{-1}^{1} = \frac{2}{3}
\]

\[
<g_2, g_0> = \int_{-1}^{1} x^4 \, dx = \frac{1}{5} x^5 \bigg|_{-1}^{1} = \frac{2}{5} \quad <f, g_0> = \int_{-1}^{1} e^{|x|} \, dx = 2 \int_{0}^{1} e^x \, dx = 2(e-1)
\]

\[
<f, g_2> = \int_{-1}^{1} x^2 e^{|x|} \, dx = 2 \int_{0}^{1} x^2 e^x \, dx = 2 e^x (x^2 - 2x + 2) \bigg|_{0}^{1} = 2e - 4
\]

\[
\begin{bmatrix}
  2 & \frac{2}{3} \\
  \frac{2}{3} & \frac{2}{5}
\end{bmatrix}
\begin{bmatrix}
  a_0 \\
  a_2
\end{bmatrix}
= \begin{bmatrix}
  2(e-1) \\
  2(e-2)
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
  a_0 \\
  a_2
\end{bmatrix}
= \begin{bmatrix}
  2(e-1) \\
  2(e-2)
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
  a_0 \\
  a_2
\end{bmatrix}
= \begin{bmatrix}
  -\frac{3}{2} e + \frac{21}{4} \\
  -\frac{15}{2} e - \frac{75}{4}
\end{bmatrix}
\]

\[
a_0 = -\frac{3}{2} e + \frac{21}{4}
\]

\[
a_2 = -\frac{15}{2} e - \frac{75}{4}
\]
Problem T4 (8 points): (1) Determine the nodes and weights for the Gaussian quadrature formula of the form
\[ \int_{-1}^{1} x^4 f(x) dx \approx A_0 f(x_0) + A_1 f(x_1). \]

We need to find Gaussian quadrature for \( n = 1 \) \( w(x) = x^4 \)
\( \{x_0, x_1\} \) are the roots of \( w \)-orthogonal polynomial of degree 2.
We first find \( w \)-orthogonal polynomials
\( P_0(x) = 1 \) \( P_1(x) = x - a_1 \)
\[ a_1 = \frac{\langle x P_0, P_0 \rangle}{\langle P_0, P_0 \rangle} = \frac{\int_{-1}^{1} x \cdot x^4 dx}{\int_{-1}^{1} x^4 dx} = 0 \]
\[ b_2 = \frac{\langle x P_1, P_0 \rangle}{\langle P_0, P_0 \rangle} = \frac{\int_{-1}^{1} x^2 \cdot x^4 dx}{\int_{-1}^{1} x^4 dx} = \frac{2}{5} \]
\[ = \frac{5}{7} \]
\[ = \frac{2}{5} \]
Continue above
Gaussian quadrature for \( n = 1 \)
is exact for polynomials of degree \( \leq 2n+1 \)
\[ 2n+1 = 3 \quad \text{Not 4} \]
\[ \text{FALSE} \]
Problem P1 (20 points): It is suspected that the high amounts of tannin in mature oak leaves inhibit the growth of the winter moth (Operophtera bromata L., Geometridae) larvae that extensively damage these trees in certain years. The following table lists the average weight of two samples of larvae at times in the first 28 days after birth. The first sample was reared on young oak leaves, whereas the second sample was reared on mature leaves from the same tree.

<table>
<thead>
<tr>
<th>Day</th>
<th>Sample 1 average weight (mg)</th>
<th>Sample 2 average weight (mg)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>6.67</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>17.33</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>42.67</td>
</tr>
<tr>
<td></td>
<td>13</td>
<td>37.33</td>
</tr>
<tr>
<td></td>
<td>17</td>
<td>30.10</td>
</tr>
</tbody>
</table>
|     | 20  | 29.31| 9.44 
|     | 28  | 28.74| 8.89 |

(a): Calculate natural cubic splines \( S(x) \) to approximate the average weight curve for each sample.

1. Let \( t_0 = 0, t_1 = 6, t_2 = 10, t_3 = 13, t_4 = 17, t_5 = 20, t_6 = 28 \). Denote \( z_i = S''(t_i) \). To construct the cubic splines for each sample, write down the linear system that you use to solve for \( z_1, z_2, \ldots, z_6 \). (4 points)

2. Graph the cubic splines for Sample 1. (3 points)
3. Graph the cubic splines for Sample 2. (3 points)
4. For Sample 1, please write down the cubic spline you get when \( 6 \leq t \leq 10 \). Compute \( S(7) \). (3 points)
5. For Sample 2, please write down the cubic spline you get when \( 6 \leq t \leq 10 \). Compute \( S(7) \). (3 points)

(b): Find an approximate maximum average weight for each sample by determining the maximum of the spline. Explain what you do. (4 points)

c): Please include your code of cubic spline construction here. You can print it or take a snapshot. You will submit your code on canvas as well.
**Problem P2 (20 points):** Write programs to evaluate $I = \int_a^b f(x)dx$ using the composite Trapezoidal rule and the composite Simpson’s rule with $n$ subdivisions, calling the result $I_n$. The error is $E_n = I - I_n$ and the ratio is

$$R_n = \frac{E_n}{E_{2n}}.$$

Fill in the following table.

1. Evaluate $\int_1^2 x \ln x dx$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$I_n$</th>
<th>$E_n$</th>
<th>Ratio</th>
<th>$I_n$</th>
<th>$E_n$</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>32</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>64</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>128</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>256</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

What can you say about the convergence of the composite Trapezoidal rule and the composite Simpson’s rule for this integration? Explain.

2. Evaluate $\int_0^2 x^2 dx$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$I_n$</th>
<th>$E_n$</th>
<th>Ratio</th>
<th>$I_n$</th>
<th>$E_n$</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>32</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>64</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>128</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>256</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

What can you say about the convergence of the composite Trapezoidal rule and the composite Simpson’s rule for this integration? Explain.

3. Please include your code of the composite Trapezoidal rule and Simpson’s rule here. You can print it or take a snapshot. You will submit your code on canvas as well.