I. Function approximation with given points.

Given points. \[ \{(x_i, y_i)\}_{i=0}^{n} \] (which are assumed to be samples of a function \( f \)). Find a function \( p \) such that:

\[ p(x_i) = y_i \quad i=0, 1, \ldots, n. \]

II. Regression in statistical learning theory.

\( x \) is sampled from a probability measure in \( \mathbb{R}^d \).

\[ y = f(x) + \xi. \]

Where \( \xi \) is a random noise independent of \( x \).

Given samples. \[ \{(x_i, y_i)\}_{i=1}^{n}, \] construct a function \( \hat{f} \).

Error \( \| \hat{f} - f \|_2 \).

Mean Square Error: \( E \| \hat{f} - f \|_2^2 \rightarrow 0 \) as \( n \rightarrow \infty \).

?? Rate of convergence.

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Interpolation. Given \( n+1 \) data points. \[ \{(x_i, y_i)\}_{i=1}^{n+1}, \] we want to express an interpolation function \( f(x) \) as a linear combination of a set of basis functions \( \{q_0, q_1, \ldots, q_n\} \) so that:

\[ f(x) \approx c_0 q_0(x) + c_1 q_1(x) + \cdots + c_n q_n(x) \]

where \( c_0, c_1, \ldots, c_n \) are to be determined.

Need:

\[ f(x_i) = c_0 q_0(x_i) + c_1 q_1(x_i) + \cdots + c_n q_n(x_i) = y_i \]

\[ i=0, 1, \ldots, n. \]
Polynomial interpolation

Natural basis. \( \phi_0(x) = 1 \), \( \phi_1(x) = x \), \( \phi_2(x) = x^2 \), \ldots, \( \phi_n(x) = x^n \)

Polynomial of degree \( \leq n \). \( P_n(x) = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n \)

Set. \( P_n(x_i) = y_i \quad i = 0, 1, \ldots, n \).

\[
\begin{bmatrix}
1 & x_0 & x_0^2 & \cdots & x_0^n \\
1 & x_1 & x_1^2 & \cdots & x_1^n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & x_n & x_n^2 & \cdots & x_n^n \\
\end{bmatrix}
\begin{bmatrix}
c_0 \\
c_1 \\
\vdots \\
c_n \\
\end{bmatrix}
= 
\begin{bmatrix}
y_0 \\
y_1 \\
\vdots \\
y_n \\
\end{bmatrix}
\]

Vandermonde matrix denoted by \( V_n \).

Thm 1 (Determinant of \( V_n \)). For a non-negative integer \( n \), and any sequence \( \{x_0, \ldots, x_n\} \) of \( n+1 \) points in \( \mathbb{R} \),

\[
\det(V_n) = \prod_{j=0}^{n-1} \prod_{k=j+1}^n (x_k - x_j).
\]

- \( V_n \) is invertible if \( \{x_0, x_1, \ldots, x_n\} \) are distinct.
- There exists a unique \( \mathbf{c}^* = [c_0^*] \) such that. \( V_n \mathbf{c}^* = \mathbf{y} \)
- There exists a unique polynomial with degree \( \leq n \) that interpolates \( \{ (x_i, y_i) \} \) if \( \{x_0, x_1, \ldots, x_n\} \) are distinct.

Thm 2. If \( \{x_0, x_1, \ldots, x_n\} \) are distinct real numbers, then for arbitrary values \( y_0, y_1, \ldots, y_n \), there is a unique polynomial \( P_n \) of degree at most \( n \) such that.

\[
P_n(x_i) = y_i \quad i = 0, 1, \ldots, n.
\]
Another proof of Theorem 2. resulting in Newton Interpolation Formula.

1. Interpolate \((x_0, y_0)\). \(P_0(x) = c_0\) \(c_0 = y_0\).

2. Interpolate \((x_0, y_0), (x_1, y_1)\). \(P_1(x) = P_0(x) + c_1(x-x_0)\). Need \(P_1(x) = P_0(x) + c_1(x-x_0)\) to choose \(c_1\) such that \(P_1(x_2) = P_1(x_2) + c_1(x_2-x_0)(x-x_1)\).

3. Interpolate \((x_0, y_0), (x_1, y_1), (x_2, y_2)\). \(P_2(x) = P_1(x) + c_2(x-x_0)(x-x_1)\). Choose \(c_2\) such that \(P_2(x_3) = P_2(x_3) + c_2(x_3-x_0)(x-x_1)(x-x_2)\).

Obtain \(P_{k-1}(x)\) \(\deg \leq k-1\) that interpolates \((x_0, y_0) \ldots (x_{k-1}, y_{k-1})\).

Construct \(P_k(x) = P_{k-1}(x) + c_k(x-x_0) \ldots (x-x_{k-1})\). \(\deg \leq k\) by choosing \(c_k\) such that \(P_k(x_k) = P_{k-1}(x_k) + c_k(x_k-x_0) \ldots (x_k-x_{k-1})\).

Then \(P_k(x)\) interpolates \((x_0, y_0) \ldots (x_k, y_k)\).

Finally, we obtain \(P_n(x)\) of \(\deg \leq n\) that interpolates \(f(x_i, y_i)\) for \(i=0\) to \(n\).

Newton form of Interpolation Polynomial

\[
\begin{align*}
P_0(x) &= c_0 \\
P_1(x) &= P_0(x) + c_1(x-x_0) \\
& \vdots \\
P_k(x) &= P_{k-1}(x) + c_k(x-x_0) \ldots (x-x_{k-1}) \\
P_n(x) &= \sum_{i=0}^{k} c_i \prod_{j=0}^{i-1} (x-x_j) \quad \text{where} \quad \prod_{j=0}^{m}(x-x_j) = 1 \quad \text{whenever} \quad m < 0
\end{align*}
\]
How to evaluate \( P_k(x) \) efficiently if the coefficients \( c_0, c_1, \ldots, c_k \) are known?

\[
P_k(x) = \sum_{i=0}^{k} c_i \prod_{j=0}^{i-1} (x-x_j).
\]

Let \( d_0 = x-x_0 \), \( d_1 = x-x_1 \), \ldots,

\[
= \sum_{i=0}^{k} c_i \prod_{j=0}^{i-1} d_j = \sum_{i=0}^{k} c_i d_0 \cdots d_{i-1}.
\]

\[
= c_0 + c_1 d_0 + c_2 d_0 d_1 + c_3 d_0 d_1 d_2 + \cdots + c_k d_0 d_1 \cdots d_{k-1}.
\]

\[
= \left( \left( c_k d_{k-1} + c_{k-1} \right) d_{k-2} + c_{k-2} d_{k-3} + \cdots + c_1 \right) d_0 + c_0.
\]

**Algorithm.**

\[
\begin{align*}
U_k & \leftarrow c_k, \\
U_{k-1} & \leftarrow U_k d_{k-1} + c_{k-1}, \\
U_{k-2} & \leftarrow U_{k-1} d_{k-2} + c_{k-2}, \quad \vdots \\
U_0 & \leftarrow U_1 d_0 + c_0.
\end{align*}
\]

How to compute the coefficients \( c_i \).

\[
c_k = \frac{y_k - P_{k-1}(x_k)}{(x_k-x_0)(x_k-x_1) \cdots (x_k-x_{k-1})}
\]
Arrange Form of the Interpolating Polynomial

\[ p(x) = y_0 \cdot l_0(x) + y_1 \cdot l_1(x) + \cdots + y_n \cdot l_n(x) = \sum_{k=0}^{n} y_k \cdot l_k(x) \]

\[ l_k(x), l_1(x), \ldots, l_n(x) \] are polynomials depending on the nodes \( x_0, x_1, \ldots, x_n \).

\[ l_i(x) = \prod_{j=0, j \neq i}^{n} \frac{x-x_j}{x_i-x_j} \quad 0 \leq i \leq n \]

For example, \[ l_0(x) = \frac{n}{\prod_{j=1}^{x} x-x_j} \frac{x-x_j}{x_0-x_j} \]

These polynomials satisfy \[ l_i(x_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases} \]

Then \[ p(x) = \sum_{k=0}^{n} y_k \cdot l_k(x_i) = y_i \], so \( p(x) \) interpolates \[ \{(x_i, y_i)\}_{i=0}^{n} \]

Example:

<table>
<thead>
<tr>
<th>( x )</th>
<th>5</th>
<th>-7</th>
<th>-6</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>1</td>
<td>-23</td>
<td>-54</td>
<td>-954</td>
</tr>
</tbody>
</table>

\( x_0 = 5 \), \( x_1 = -7 \), \( x_2 = -6 \), \( x_3 = 0 \).

\[ l_0(x) = \frac{(x+1)(x+6)x}{(5+7)(5+6)} = \frac{1}{660} x(x+6)(x+7) \]

\[ l_1(x) = \frac{(x-5)(x+6)x}{(-7-5)(-7+6)(-7)} = -\frac{1}{84} x(x-5)(x+6) \]

\[ l_2(x) = \frac{(x-5)(x+7)x}{(-6-5)(-6+7)(-6)} = -\frac{1}{66} x(x-5)(x+7) \]

\[ l_3(x) = \frac{(x-5)(x+7)(x+6)}{(-5)(7)(6)} = -\frac{1}{210} (x-5)(x+6)(x+7) \]

Interpolating polynomial \[ p(x) = l_0(x) - 23l_1(x) - 54l_2(x) - 954l_3(x) \]