Three polynomial interpolation forms.

I. Monomial basis, resulting a linear system of Vandermonde matrices.
   - The Vandermonde matrix is often ill-conditioned.
   - Not suitable for computation.

II. Newton form.
   - Best for computation.
   - If more data points are added to the interpolation problem, the coefficients already computed will not have to be changed. Easily accommodates additional data to be interpolated.

III. Lagrangean form.

The Error in Polynomial Interpolation.

\( f: \text{ a function on } [a, b] \)
\( \{(x_i, y_i)\}_{i=0}^n: \text{ n samples of the function such that } y_i = f(x_i) \)
\( p(x) \text{ the polynomial of degree } \leq n \) that interpolates \( f(x_i, y_i) \).

**Question:** \( |f(x) - p(x)| \) error.

**Theorem 3.** [Theorem on Polynomial Interpolation Error]

Let \( f \) be a function in \( C^{n+1}[a, b] \) and let \( p \) be the polynomial of degree at most \( n \) that interpolates the function \( f \) at \( n+1 \) distinct points \( x_0, x_1, \ldots, x_n \) in the interval \([a, b]\). To each \( x \in [a, b] \), there corresponds a point \( s_x \in [a, b] \) such that:

\[
  f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(s_x) \sum_{i=0}^{\frac{n}{i}} (x-x_i).
\]
Proof.

If \( x = x_0 \) or \( x_1 \) or \( \cdots \) \( x_n \), Eq. 10 is obviously true.

Assume \( x \) is any point other than a node. Let \( x \) be fixed.

Define \( \phi(t) = \sum_{i=0}^{n} (t - x_i) \), \( \omega(x) = \prod_{i=0}^{n} (x - x_i) \).

\[ \phi := f - P - \lambda x w \quad \lambda x \text{ is set such that} \]
\[ \lambda x = \frac{f(x) - p(x)}{\omega(x)} \]

\( \phi \in C^{n+1} [a,b] \). \( \phi(x) = 0 \). \( \phi(x_0) = 0 \) \( \cdots \) \( \phi(x_n) = 0 \).

Rolle's theorem. \( \phi' \) has at least \( n+1 \) distinct zeros in \((0,b)\).

\[ \phi'' - \cdots \cdots - n \]

\[ \phi^{(n+1)} \]

\[ 1 \text{ zero is } (a,b) \]

Let it be \( 5x \).

\[ \phi^{(n+1)}(5x) = \phi^{(n+1)}(5x) - \phi^{(n+1)}(5x) - \lambda x \omega^{(n+1)}(5x) \]

\[ = f^{(n+1)}(5x) - \lambda x (n+1)! \Rightarrow \lambda x = \frac{f^{(n+1)}(5x)}{(n+1)!} \]

Example. \( f(x) = \sin x \) is approximated by a polynomial of degree 9 that interpolates \( f \) at 10 points in the interval \([0,1]\).

How large is the error?

\[ |\sin x - p(x)| \leq \frac{1}{10!} \left| f^{(10)}(5x) \right| \]

\[ \prod_{i=0}^{9} (x - x_i) \leq \frac{1}{10!} \]

\[ |f^{(10)}(x)| \leq 1 \left| \prod_{i=0}^{9} (x - x_i) \right| \leq 1 \]
Chebyshev polynomials

Chebyshev polynomials are defined recursively.

\[ T_0(x) = 1 \quad T_1(x) = x \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \]

Such that:

\[ T_2(x) = 2x^2 - 1 \]
\[ T_3(x) = 4x^3 - 3x \]
\[ T_4(x) = 8x^4 - 8x^2 + 1 \]
\[ T_5(x) = 16x^5 - 20x^3 + 5x \]

Closed form of Chebyshev polynomials for \( x \in [-1, 1] \).

Theorem 4. For \( x \in [-1, 1] \), the Chebyshev polynomials have the closed-form expression:

\[ T_n(x) = \cos(n \cos^{-1} x) \quad n \geq 0 \]

Proof: Recall.

\[ \cos(A + B) = \cos A \cos B - \sin A \sin B \]

Then:

\[ \cos(n+1) \theta = \cos \theta \cos n \theta - \sin \theta \sin n \theta \]
\[ \cos(n-1) \theta = \cos \theta \cos n \theta + \sin \theta \sin n \theta \]

Adding these two equations gives:

\[ \cos(n+1) \theta + \cos(n-1) \theta = 2 \cos \theta \cos n \theta \]

\[ \Rightarrow \cos(n+1) \theta = 2 \cos \theta \cos n \theta - \cos(n-1) \theta \]

Let \( x = \cos \theta \). \( f_0(x) = \cos \theta = \cos(n \cos^{-1} x) \)

\[ f_1(x) = x \]
\[ f_{n+1}(x) = 2x f_n(x) - f_{n-1}(x) \]

So \( f_n(x) = T_n(x) \)
Properties of Chebyshev polynomials.

1. \( |T_n(x)| \leq 1 \quad x \in [-1, 1] \)

2. \( T_n(\cos \frac{j \pi}{n}) = (-1)^j \quad 0 \leq j \leq n \)

\[ T_n(\cos \frac{2j-1}{2n} \pi) = 0 \quad 1 \leq j \leq n. \]

3. \( T_n(x) = 2^{n-1} \prod_{j=1}^{n} \left[ x - \cos \frac{2j-1}{2n} \pi \right] \).

The Chebyshev polynomial, \( T_n(x) \), is of degree \( n \) and the highest order term is \( 2^{n-1} x^n \).

It has \( n \) distinct roots, \( \cos \frac{2j-1}{2n} \pi \).
Approximation error by infinity norm.

\[ \|f - p\|_\infty = \max_{x \in [a, b]} |f(x) - p(x)| \]

According to Theorem 3, \( |f(x) - p(x)| = \frac{1}{(n+1)!} \left| f^{(n+1)}(\xi) \right| \left| \prod_{i=0}^{n} (x-x_i) \right| \)

If \( x \in [-1, 1] \)

\[ \|f - p\|_\infty \leq \frac{1}{(n+1)!} \max_{x \in [a, b]} \left| f^{(n+1)}(\xi) \right| \max_{x \in [-1, 1]} \left| \prod_{i=0}^{n} (x-x_i) \right| \]

\[ \|f - p\|_\infty \leq \frac{1}{(n+1)!} \max_{x \in [-1, 1]} \left| f^{(n+1)}(\xi) \right| \max_{x \in [-1, 1]} \left| \prod_{i=0}^{n} (x-x_i) \right| \]

**Theorem on Mono polynomials.**

A mono polynomial is one in which the term of highest degree has a coefficient of unity.

**Theorem 4.** If \( p \) is a mono polynomial of degree \( n \), then

\[ \|p\|_\infty = \max_{x \in [-1, 1]} |p(x)| \geq 2^{1-n}. \]

**More analysis.**

Let \( Q_{n+1}(x) = \prod_{i=0}^{n} (x-x_i) \)

\[ \|Q_{n+1}\|_\infty = \max_{x \in [-1, 1]} |Q_{n+1}(x)| \]

Then \( \|Q_{n+1}\|_\infty \geq 2^{1-(n+1)} = 2^{-n} \) by Theorem 4.

What nodes \( \{x_0, x_1, \ldots, x_n\} \) give rise to the minimum \( \|Q_{n+1}\|_\infty \)?

- \( 2^{-n} T_{n+1} \) is a mono polynomial of degree \( n+1 \)
- \( \max_{x \in [-1, 1]} |T_{n+1}(x)| = 1 \)
\[
\max_{x \in [-1,1]} \left| 2^{-n} T_{n+1}(x) \right| = 2^{-n}
\]

\[
\min_{x_0, x_1, \ldots, x_n \in [-1,1]} \| Q_{n+1} \|_{\infty} = \min_{x \in [-1,1]} \left| 2^{-n} T_{n+1}(x) \right|
\]

\[
= \min_{x \in [-1,1]} \left| \frac{n}{\Pi_{i=0}^{n} (x - x_i^c)} \right|
\]

where \( x_i^c = \cos \left( \frac{2i+1}{2n+2} \pi \right) \)

are the roots of \( T_{n+1} \).

In other words, \( \| Q_{n+1} \|_{\infty} \) is minimized if the nodes \( x_0, \ldots, x_n \)
are taken to be the roots of \( T_{n+1} \).

**Chebyshev nodes:** the roots of the Chebyshev polynomials.

**Theorem 5.** If the nodes \( x_0, x_1, \ldots, x_n \) are the roots of the
Chebyshev polynomial \( T_{n+1} \). Let \( f \) be a function in \( C^{n+1} [-1,1] \)
and let \( p \) be the polynomial of degree \( n \) that interpolates \( f \)
at \( x_0, \ldots, x_n \). Then,

\[
\| f - p \|_{\infty} = \sup_{x \in [-1,1]} |f(x) - p(x)| \leq \frac{1}{2^n (n+1)!} \max_{x \in [-1,1]} |f^{(n+1)}(x)|
\]

**Remark:** Chebyshev polynomials and Chebyshev nodes are also defined
on \([a, b]\) as well. Use the map:

\[
x = \frac{1}{2} (b-a) t + \frac{1}{2} (a+b) \quad t \in [-1,1]
\]
on equivalently,

\[
t = \frac{2}{b-a} \left[ x - \frac{1}{2} (a+b) \right] \quad x \in [a, b]
\]
If \( f \in C^{n+1} [a,b] \), and \( x_0, x_1, \ldots, x_n \) are \( n+1 \) Chebyshev nodes on \([a,b]\). Let \( p \) be the polynomial of degree \( n \) that interpolates \( f \) at \( x_0, x_1, \ldots, x_n \). Then,

\[
\| f - p \|_{\infty} = \sup_{x \in [a,b]} |f(x) - p(x)| \leq \frac{1}{2n(n+1)!} \left( \frac{b-a}{2} \right)^{n+1} \| f^{(n+1)} \|_{\infty}.
\]

**Uniform convergence**

\( \sqrt[n]{n} \) interpolating polynomial of degree \( n \).

**Expectation**. \( \| f - P_n \|_{\infty} = \max_{x \in [a,b]} |f(x) - P_n(x)| \to 0 \) as \( n \to \infty \).

- The result may depend on the interpolation nodes.
- If Chebyshev nodes are used, \( \| f - P_n \|_{\infty} \to 0 \) if \( \| f^{(n)} \|_{\infty} \leq M \) for any \( n \).

**Example: Runge example**

\( f(x) = \frac{1}{1 + 25x^2} \), \( x \in [-1, 1] \) nodes = equally spaced nodes.

**Nodes**: \( x_i = \frac{2i}{n} - 1 \), \( i = 0, 1, \ldots, n \).

\( P_n(x) \): Interpolating polynomials of deg \( \leq n \).

\[
\| f - P_n \|_{\infty} = \max_{x \in [-1,1]} |f(x) - P_n(x)| \to \infty \quad \text{as} \quad n \to \infty.
\]

**Why?**

\[
\max_{x \in [-1,1]} |f(x) - P_n(x)| \leq \max_{x \in [-1,1]} \left| \frac{f^{(n+1)}(x)}{(n+1)!} \right| \max_{x \in [-1,1]} \frac{1}{x-x_i}.
\]

\[
Q_{n+1}(x) = (x-x_0) \cdots (x-x_n)
\]

\[
\max_{x \in [-1,1]} \left| Q_{n+1}(x) \right| \leq n! \cdot \frac{h^{n+1}}{n!} = \frac{1}{n+1}.
\]
\[ \| f - P_n \|_{\infty} \leq \frac{n! h^{n+1}}{(n+1)!} \max_{x \in [1,1]} |f^{(n+1)}(x)| = \frac{Mn h^{n+1}}{n+1} \]

Problem: \( M_n \) increases when \( n \) increases, very fast.

However, if \( P_n \) are interpolated at Chebyshev nodes, \( \| f - P_n \|_{\infty} \to 0 \) as \( n \to \infty \).

Classical results.

Thm. Faber's Theorem.

For any prescribed system of nodes,
\[ a \leq x_0^{(m)} < x_1^{(m)} < \ldots < x_n^{(m)} \leq b \quad n \geq 0 \]
there exists a continuous function \( f \) on \([a,b]\) such that the interpolating polynomials, for \( f \) using these nodes, fail to converge uniformly to \( f \).

Thm 2. If \( f \) is a continuous function on \([a,b]\), then there is a system of nodes such that the interpolating polynomials for \( f \) at these nodes satisfy \( \lim_{n \to \infty} \| f - P_n \|_{\infty} = 0 \).

Theorem 2 is obtained by joining the Weierstrass Approximation Thm. and the Chebyshev Alternation Thm.
Weierstrass Approximation Theorem

If \( f \) is continuous on \([a, b]\) and if \( \varepsilon > 0 \), then there is a polynomial \( P \) satisfying \( |f(x) - px)\) \( \leq \varepsilon \) on the interval \([a, b]\).

Proof. One can prove it on \([0, 1]\) without loss of generality.

On \([0, 1]\), \( f \in C[0, 1] \), construct a sequence of Bernstein polynomials \( B_n f \) that converges to \( f \) uniformly.

Bernstein Polynomials:

\[
(B_n f)(x) = \sum_{k=0}^{n} \binom{n}{k} f \left( \frac{k}{n} \right) g_{nk}(x)
\]

\[
g_{nk}(x) = \binom{n}{k} x^k (1-x)^{n-k}
\]

\( B_n \) is a linear operator: \( B_n(af + bg) = aB_nf + bB_ng \).

Detailed proof is in Kincaid & Cheney.