# On Minimal Sets of Scalar Parabolic Equations with Skew-product Structures

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<u>Abstract:</u> Skew product semiflow $\Pi_t : X \times Y \to X \times Y$ which is generated by

$$\begin{cases} u_t = u_{xx} + f(y \cdot t, x, u, u_x), \quad t > 0, \quad 0 < x < 1, \quad y \in Y, \\ \\ D \quad and \quad N \quad boundary \quad conditions \end{cases}$$

is considered, where $X$ is an appropriate subspace of $H^2(0,1)$, $(Y, \mathbb{R})$ is a minimal flow with compact phase space. It is shown that a minimal set $E \subset X \times Y$ of $\Pi_t$ is an almost 1-1 extension of $Y$, that is, set $Y_0 = \{y \in Y | card(E \cap P^{-1}(y)) = 1\}$ is a residual subset of $Y$, where $P : X \times Y \to Y$ is the natural projection. Consequently, if $(Y, \mathbb{R})$ is almost periodic minimal, then any minimal set $E \subset X \times Y$ of $\Pi_t$ is an almost automorphic minimal set. It is also proved that dynamics of $\Pi_t$ is closed in the category of almost automorphy, that is, a minimal set $E \subset X \times Y$ of $\Pi_t$ is almost automorphic minimal if and only if $(Y, \mathbb{R})$ is almost automorphic minimal. Asymptotically almost periodic parabolic equations and certain coupled parabolic systems are discussed. Examples of non-almost periodic almost automorphic minimal sets are provided.

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## 1. Introduction

In the current paper, we shall study fundamental dynamical issues for a family of scalar parabolic equations

$$u_t = u_{xx} + f(y \cdot t, x, u, u_x), \quad t > 0, \quad 0 < x < 1, \qquad (1.1)_y$$

with boundary conditions:

$$\beta u(t, 0) + (1 - \beta) u_x(t, 0) = 0, \beta u(t, 1) + (1 - \beta) u_x(t, 1) = 0, t > 0, \qquad (1.2)$$

where $\beta = 0$ or $1$, $y \in Y$, $(Y, \mathbb{R})$ is a minimal flow with compact metric phase space $Y$, and the function $f(y, x, u, p)$ is $C^2$ in $x$, $u$, $p$, and Lipschitz in $y$.

Let $X$ be a fractional power space associated with the operator $u \to -u_{xx} : \mathcal{D} \to L^2(0, 1)$ that satisfies $X \hookrightarrow C^1[0, 1]$, where $\mathcal{D} = \{u | u \in H^2(0, 1), u \text{ satisfies } (1.2)\}$ ([21]). It is well known that for any $y \in Y$, $U \in X$, equation $(1.1)_y$-$(1.2)$ admits a (locally) unique solution $u(t, \cdot, U, y)$ in $X$ with $u(0, \cdot, U, y) = U(\cdot)$. Thus equations $(1.1)_y$-$(1.2)$ $(y \in Y)$ generate a (local) skew-product semi-flow $\Pi_t$ on $X \times Y$,

$$\Pi_t(U, y) = (u(t, \cdot, U, y), y \cdot t), \quad t > 0. \qquad (1.3)$$

It is important to note that the (local) skew-product semi-flow (1.3) restricted to an $\omega$-limit set or a minimal invariant set defines a usual skew-product (two-sided) flow ([20], [21]).

One of the fundamental dynamical issues in the study of a skew-product flow is the lifting properties of minimal sets (for general discussion, see [15], [33], [35], and references therein). In terms of differential equations, the lifting properties describe whether or how much certain structure (say, periodicity, almost periodicity) in an equation can be lifted to its solutions. A lifting property is also closely related to the asymptotic behavior of bounded solutions of a differential equation (see Section 4 and discussion below). Before stating the main results of the current paper, we first introduce some terminology.

Since only real flows are considered in the current paper, without causing confusion, we shall simply denote a flow $(E, \mathbb{R})$ as $E$. Let $p : E \to Y$ be a homomorphism of minimal flows (that is, $p : E \to Y$ is continuous and surjective, and $p(x \cdot t) = p(x) \cdot t$ for any $x \in E$, $t \in \mathbb{R}$, see [39], [40]). Assume that $E$ is a compact metric space. $E$ is a <u>1-1 extension</u> of $Y$ if $card E \cap p^{-1}(y) = 1$ for all $y \in Y$. $E$ is an

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almost 1-1 extension of $Y$ if there is a residual subset $Y_0 \subset Y$ such that $card E \cap p^{-1}(y_0) = 1$ for $y_0 \in Y_0$. Note that a 1-1 extension of $Y$ is topologically conjugate to $Y$. A 1-1 extension (almost 1-1 extension) of an almost periodic minimal flow $Y$ (see section 3) is referred to as an almost periodic (almost automorphic) extension of $Y$. The notion of almost automorphy was first introduced in dynamical systems by S. Bochner ([4]). An almost automorphic function (see Section 3) is the pointwise limit of a 'jointly almost automorphic' net of (Bohr) almost periodic functions ([39]). It has many properties in common with the usual (Bohr) almost periodic function, for example, it possesses (non-unique) Fourier series in a deterministic way ([39]). An almost periodic function, is necessarily almost automorphic, but the converse is not true (see [41]). The connection between the almost automorphy and the almost periodicity is indicated in [39] as follows: a function is (Bohr) almost periodic if and only if its compact hull (with compact open topology) consists of almost automorphic functions. Unlike the almost periodic case, the compact hull of a continuous (non-almost periodic) almost automorphic function contains only residually many almost automorphic functions (see Section 3 or [39] for details). We refer to the closure of an almost automorphic (almost periodic) motion in a dynamical system as an almost automorphic (almost periodic) minimal set (see section 3). It is known that that a flow $E$ is an almost automorphic minimal flow if and only if it is an almost automorphic extension of an almost periodic minimal flow $Y$, and the almost automorphic points on $E$ (points $x \in E$ such that $x \cdot t$ is almost automorphic) are precisely those $x$ with $E \cap p^{-1}(y) = x$ for some $y \in Y$ (see Section 3 or [39]). Thus, an almost automorphic minimal set contains residually many almost automorphic points (an almost periodic minimal set consists of almost periodic hence almost automorphic points).

In the study of scalar almost periodic equations $(1.1)_y$-$(1.2)$ (that is, $Y$ in $(1.1)_y$-$(1.2)$ is almost periodic minimal) as well as almost periodic scalar ODEs (note that scalar ODEs form sub-dynamical systems of $(1.1)_y$-$(1.2)$ with $\beta = 0$), the existence of almost automorphic solutions is crucial. There are many examples (see [18], [24] and Section 5 of the current paper) in which a scalar almost periodic ODE admits almost automorphic solutions but no almost periodic solutions. In other word, in such systems, there is a minimal set which is almost automorphic but not almost periodic. In the current paper, we shall show that 'almost automorphy' is essential for dynamical system $(1.1)_y$-$(1.2)$ or

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(1.3). Let $P : X \times Y \to Y$ be the natural projection. Then for any minimal set $E \subset X \times Y$ of (1.3), $P : E \to Y$ clearly defines a homomorphism of minimal flows. Our main results are as follows.

**Main Theorem:** *1) Any minimal set $E$ of (1.3) is an almost* 1-1 *extension of $Y$.*

*2) A minimal set $E$ of (1.3) is almost automorphic minimal if and only if $Y$ is almost automorphic minimal.*

We conclude from 1) that any minimal set of (1.3) is an almost automorphic extension of $Y$ if $Y$ is almost periodic minimal. This solves a problem left open in [37].

There are two important facts involved in our main results. First, the result 1) is optimal as is suggested by examples in section 5. Second, by the result 2), although the lifting property is not 'closed' in the category of almost periodicity, it is however 'closed' in the category of almost automorphy.

We would like to point out that there are major differences between the periodic case and the almost periodic case in the study of $(1.1)_y$-(1.2) or (1.3). In the case that the minimal flow $Y$ in $(1.1)_y$-(1.2) is periodic (that is, $Y$ is minimal and any motion in $Y$ is periodic), it is well known that each minimal set of (1.3) is a precise 1-1 extension of $Y$, namely, a periodic minimal set (see [5], [9]). In fact, much more can be said in the case of periodic dependence. For example, each $\omega$-limit set of (1.3) is necessarily a periodic minimal set (a 1-1 extension of $Y$), therefore, each positively bounded motion in (1.3) is asymptotically periodic ([5], [9]). Similar results are no longer true in the almost periodic case. It is shown in [37] for the almost periodic case that each $\omega$-limit set of (1.3) may contain two but no more than two minimal sets, moreover, a positively bounded motion $\Pi_t(U_0, y_0)$ of (1.3) is asymptotically almost periodic if and only if its $\omega$-limit set $\omega(U_0, y_0)$ is an almost periodic extension of $Y$ (see [37] or section 4). Thus, asymptotic almost periodicity can not be always expected for positively bounded solutions of (1.3). One often observes a kind of 'non-uniform' asymptotic phenomenon in (1.3) if an (non-almost periodic) almost automorphic motion appears.

To explain this, we note that an alternative way to state the asymptotic almost periodicity is the following: $\Pi_t(U_0, y_0)$ is asymptotically almost periodic if and only if $\omega(U_0, y_0)$ is minimal and every point in $\omega(U_0, y_0)$ is an almost automorphic point. In general, if $\omega(U_0, y_0)$ is a non-almost periodic minimal set (hence an almost automorphic extension

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of $Y$), then one cannot conclude that $\Pi_t(U_0, y_0)$ is asymptotically almost automorphic (this is unlikely to be true even $\omega(U_0, y_0) \cap P^{-1}(y_0)$ is a single almost automorphic point). Instead, $\Pi_t(U_0, y_0)$ is 'almost' asymptotically almost automorphic if $\omega(U_0, y_0)$ is minimal since in this case $\omega(U_0, y_0)$ contains residually many almost automorphic points.

The paper is organized as follows. We prove our main result 1) in Section 2 by using the properties of the zero number developed in [1], [8], [27]. Cases in which a minimal set $E$ is a precise 1-1 extension of $Y$ are also discussed in Section 2. In Section 3, we give our attention to cases in which $Y$ is either an almost automorphic or an almost periodic minimal flow, and prove the main result 2). Some applications of our current work to nonautonomous parabolic equations (for example, almost periodic equations, asymptotically almost periodic equations, and certain recursively coupled systems) are provided in Section 4. In Section 5, we discuss two examples of almost periodic scalar ODEs from Fink [18] and Johnson [24].

We remark that the results in this paper also hold true for equation $(1.1)_y$ with more general boundary conditions (e.g. third boundary conditions).

Dynamical studies of scalar parabolic equations in one space dimension go back to [6], [28], [29] for autonomous equations and are developed later in [2], [5], [7], [9], [10], [17], [34] for periodic dependent equations. Almost periodic dependent cases are discussed in [37], [38]. In [38], work of [11], [12] on Floquet theory is applied. For dynamical studies of parabolic equations in higher space dimensions, see [13], [14], [30], [31], [36], [42], [43] and references therein.

## 2. Lifting Properties of Minimal Sets

The main subject of this section is to study lifting properties for minimal invariant sets of (1.3). We note that any minimal set of (1.3) is compact and the flow $\Pi_t$ restricted to it defines a global (two-sided) flow ([20]).

**Definition 2.1.** *For a given $C^1$ function $u : [0, 1] \to \mathbb{R}^1$, the zero number $z(u(\cdot))$ of $u$ is defined as*

$$z(u(\cdot)) = \#\{x \in (0, 1) | u(x) = 0\}.$$

The following Lemma is proved in [8] (see also [1], [27] for cases with stronger conditions).

**Lemma 2.1.** *Consider the following scalar linear parabolic equation:*

$$\begin{cases} u_t = a(t, x)u_{xx} + b(t, x)u_x + c(t, x)u, \quad t > 0, \quad x \in (0, 1), \\ \beta u(t, 0) + (1 - \beta)u_x(t, 0) = 0, \beta u(t, 1) + (1 - \beta)u_x(t, 1) = 0, t > 0, \end{cases} \tag{2.1}$$

*where $a$, $a_t$, $a_x$, $b$ and $c$ are bounded continuous functions, $a \geq \delta > 0$. Let $u(t, x)$ be a classical nontrivial solution of (2.1). Then the following holds:*

*1) $Z(u(t, \cdot))$ is finite for $t > 0$ and is nonincreasing in $t$;*

*2) $Z(u(t, \cdot))$ can drop only at $t_0$ such that $u(t_0, \cdot)$ has a multiple zero in $[0, 1]$;*

*3) $Z(u(t, \cdot))$ can drop only finite many times, and there exists a $t^* > 0$ such that $u(t, \cdot)$ has only simple zeros in $[0, 1]$ as $t \geq t^*$ (hence $Z(u(t, \cdot)) = $ constant as $t \geq t^*$).*

Our next lemma can also be found in [37] or [40].

**Lemma 2.2.** *Let $p : E \to Y$ be homomorphism of minimal flows, where $E, Y$ are compact $T_2$. Then there is a residual subset $Y_0 \subset Y$ which is invariant and satisfies the following properties:*

*1) For any $y_* \in Y_0$, $y \in Y$ and any $x_* \in p^{-1}(y_*)$, if $\{t_n\}$ is a sequence with $t_n \to +\infty$ or $-\infty$ such that $y \cdot t_n \to y_*$ as $n \to \infty$, then there is a sequence $\{x_n\} \subset p^{-1}(y)$ such that $x_n \cdot t_n \to x_*$ as $n \to \infty$.*

*2) Consider $q : Y \to 2^E$, $y \mapsto p^{-1}(y)$. If $E$ is compact metric and $2^E$ is furnished with Hausdroff metric $\sigma$, then $Y_0$ is precisely the set of continuous points of $q$.*

**Proof.** 1) Consider the set valued map $q : Y \to 2^E$, $y \mapsto p^{-1}(y)$. It is easy to see that $q$ is upper semi-continuous. Let $Y_0 \subset Y$ be the set of continuous points of $q$. Then $Y_0 \subset Y$ is a residual subset ([3]). $Y_0$ is clearly invariant. Now, 1) follows immediately from the definition of lower semi-continuity for set valued maps ([3]).

2) Recall, for any $E_1$, $E_2 \in 2^E$, $\sigma(E_1, E_2) = \max\{\mu(E_1, E_2), \mu(E_2, E_1)\}$, where $\mu(E_1, E_2) = \max_{x \in E_1} \min_{x' \in E_2} d_E(x, x')$, $d_E$ is the metric on $E$. 2) is an easy consequence of 1). ∎

**Definition 2.2** *Let $X$, $Y$ be as in (1.3) and let $P : X \times Y \to Y$ be the natural projection. For any $y \in Y$, we define relation '$\geq$' ('$>$') on $P^{-1}(y)$ as follows. For any $(U_1, y)$, $(U_2, y) \in P^{-1}(y)$, $(U_1, y) \geq (>)(U_2, y)$ if there is $T > 0$ such that*

$$(1 - \beta)u(t, 0, U_1, y) + \beta u_x(t, 0, U_1, y) \geq (>)(1 - \beta)u(t, 0, U_2, y) + \beta u_x(t, 0, U_2, y)$$

*for all $t \geq T$.*

**Lemma 2.3.** *For any $y \in Y$, '$\geq$' defines a total ordering on $P^{-1}(y)$, i.e. '$\geq$' is a partial ordering of $P^{-1}(y)$ and any $(U_1, y)$, $(U_2, y) \in P^{-1}(y)$ are comparable.*

**Proof.** For any $y \in Y$ and $(U_1, y)$, $(U_2, y) \in P^{-1}(y)$, we note that $V(t, x, y) = u(t, x, U_1, y) - u(t, x, U_2, y)$ satisfies a linear parabolic equation of form (2.1) ([5], [37]). Therefore, by Lemma 2.1, if $U_1 \neq U_2$, then there is a $T > 0$ such that $(1 - \beta)(u(t, 0, U_1, y) - u(t, 0, U_2, y)) + \beta(u_x(t, 0, U_1, y) - u_x(t, 0, U_2, y))$ has constant sign for $t \geq T$. The rest of the proof is straightforward. ∎

**Definition 2.3.** *For any $y \in Y$, let $A(y) \subset P^{-1}(y)$ be a subset such that $\operatorname{card} A(y) \geq 2$. For any $(U_1, y)$, $(U_2, y) \in A(y)$, $U_1 \neq U_2$, define*

$$Z_+(U_1, U_2, y) = \min_{t \geq 0} Z(u(t, \cdot, U_1, y) - u(t, \cdot, U_2, y)), \tag{2.2}$$

*and*

$$N_0(A(y)) = \min\{Z_+(U_1, U_2, y) | (U_1, y), (U_2, y) \in A(y), U_1 \neq U_2\}. \tag{2.3}$$

*For any $(U_0, y) \in A(y)$, define*

$$A(U_0, y) = \{(U, y) \in A(y) | U \neq U_0, Z_+(U, U_0, y) = N_0(A(y))\}. \tag{2.4}$$

The above quantities are well defined by Lemma 2.1.

**Lemma 2.4.** *Let $E \subset X \times Y$ be a minimal set of (1.3), $Y_0 \subset Y$ be the residual subset in Lemma 2.2, and $A(y) = E \cap P^{-1}(y)$ $(y \in Y)$. If $y \in Y_0$ is such that $\operatorname{card} A(y) \geq 2$, then $A(U_0, y)$ defined in (2.4) is non-empty for any $(U_0, y) \in A(y)$.*

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**Proof.** Take $y \in Y_0$ so that $cardA(y) \geq 2$. Let $(U_1^*, y)$, $(U_2^*, y) \in A(y)$, $U_1^* \neq U_2^*$ be such that $Z_+(U_1^*, U_2^*, y) = N_0(A(y))$. Choose a $T > 0$ such that $u(t, \cdot, U_1^*, y) - u(t, \cdot, U_2^*, y)$ has only simple zeros in $[0, 1]$ as $t \geq T$. This implies that $Z(u(t, \cdot, U_1^*, y) - u(t, \cdot, U_2^*, y)) = N_0(A(y))$ for $t \geq T$. Now for any $(U_0, y) \in A(y)$, by Lemma 2.2, there are sequences $t_n \to \infty$ and $\{U_n\} \subset A(y)$ such that $y \cdot t_n \to y$, $\Pi_{t_n}(U_0, y) \to (U_1^*, y)$, and $\Pi_{t_n}(U_n, y) \to (U_2^*, y)$ as $n \to \infty$. It follows that there exists $n_0 > 0$ such that $Z(u(t_n + T, \cdot, U_0, y) - u(t_n + T, \cdot, U_n, y)) = N_0(A(y))$ for $n \geq n_0$. Since $N_0(A(y))$ is the smallest zero number on $A(y)$, one has that $Z_+(U_0, U_n, y) = N_0(A(y))$ for $n \geq n_0$. Clearly, $U_n \neq U_0$ for $n \geq n_0$. This proves the Lemma. ∎

**Lemma 2.5.** *Let $E$, $Y_0$, $A(y)$ be as in Lemma 2.4. Then for any $y \in Y_0$ with $cardA(y) \geq 2$, there is a $(U_0, y) \in A(y)$ such that $(U_0, y) \geq (U, y)$ for any $(U, y) \in A(U_0, y)$.*

**Proof.** Suppose that the Lemma is not true. Then there is a $y^* \in Y_0$ with $cardA(y^*) \geq 2$, and, for any $(U^*, y^*) \in A(y^*)$ there is a $(U_0^*, y^*) \in A(U^*, y)$ such that $(U^*, y^*) < (U_0^*, y^*)$. Since $Z_+(U_0^*, U^*, y^*) = N_0(A(y^*))$, by Lemma 2.1, there is a $T > 0$ such that $u(t, \cdot, U_0^*, y) - u(t, \cdot, U^*, y^*)$ has only simple zeros in $[0, 1]$ and $Z(u(t, \cdot, U_0^*, y^*) - u(t, \cdot, U^*, y^*)) = N_0(A(y^*))$ for all $t \geq T$. In particular, $u(T, \cdot, U_0^*, y^*) - u(T, \cdot, U^*, y^*)$ has only simple zeros in $[0, 1]$. It follows that there is an open neighborhood $B(U^*, y^*)$ of $(U^*, y^*)$ in $A(y^*)$ such that $u(T, \cdot, U_0^*, y^*) - u(T, \cdot, U, y^*)$ has only simple zeros in $[0, 1]$ and $Z(u(T, \cdot, U_0^*, y^*) - u(T, \cdot, U, y^*)) = N_0(A(y))$ for any $(U, y^*) \in B(U^*, y^*)$. Without loss of generality, we also assume that

$$(1 - \beta)|u(T, 0, U^*, y^*) - u(T, 0, U, y^*)| + \beta|u_x(T, 0, U^*, y^*) - u_x(T, 0, U, y^*)|$$
$$< \frac{1}{2}\left[(1 - \beta)(u(T, 0, U_0^*, y^*) - u(T, 0, U^*, y^*)) + \beta(u_x(T, 0, U_0^*, y^*) - u_x(T, 0, U^*, y^*))\right]$$
$$(2.5)$$

for any $(U, y^*) \in B(U^*, y^*)$. This implies

$$(1 - \beta)(u(T, 0, U_0^*, y^*) - u(T, 0, U, y^*)) + \beta(u_x(T, 0, U_0^*, y^*) - u_x(T, 0, U, y^*)) > 0 \quad (2.6)$$

for any $(U, y^*) \in B(U^*, y^*)$. Since $N_0(A(y^*))$ is the smallest zero number on $A(y^*)$, it is clear that $u(t, \cdot, U_0^*, y^*) - u(t, \cdot, U, y^*)$ has only simple zeros in $[0, 1]$, and

$$Z(u(t, \cdot, U_0^*, y^*) - u(t, \cdot, U, y^*)) = N_0(A(y^*))$$

for any $t \geq T$ and $(U, y^*) \in B(U^*, y^*)$. Combinning the above with (2.6), one has that $(U, y^*) < (U_0^*, y^*)$ for any $(U, y^*) \in B(U^*, y^*)$, that is, $B(U^*, y^*)$ has an upper bound $(U_0^*, y^*)$ in $A(y^*)$. Note that $\{B(U^*, y^*) : (U^*, y^*) \in A(y^*)\}$ forms an open covering of the compact set $A(y^*)$. Hence there are $(U_i^*, y^*) \in A(y^*)$, $i = 1, 2, \cdots, n_0$, for some $n_0 \geq 1$, such that $\cup_{i=1}^{n_0} B(U_i^*, y^*) = A(y^*)$. Let $(U_0, y^*) \in A(y^*)$ be the maximum element of the chain $\{(U_{01}^*, y^*), \cdots, (U_{0n_0}^*, y^*)\}$ (i.e. $(U_0, y^*) \geq (U_{0i}^*, y^*)$, $i = 1, 2, \cdots, n_0$), where $(U_{0i}^*, y^*) \in A(y^*)$, $i = 1, 2, \cdots, n_0$, are upper bounds of $B(U_i^*, y^*)$. Then $(U_0, y^*)$ is a maximum element of $A(y^*)$. Now, by Lemma 2.4, $A(U_0, y^*)$ is non-empty. Clearly, $(U_0, y^*)$ is also an upper bound of $A(U_0, y^*)$, that is, for any $(U, y^*) \in A(U_0, y^*)$, $(U_0, y^*) > (U, y^*)$. This is a contradiction. ■

**Theorem 2.6.** *Any minimal set $E$ of (1.3) is an almost 1-1 extension of $Y$, that is, there is a residual set $Y_0 \subset Y$ such that $card(E \cap P^{-1}(y)) = 1$ for all $y \in Y_0$.*

**Proof.** Let $E \subset X \times Y$ be a minimal set of (1.3) and $Y_0 \subset Y$ be the residual set in Lemma 2.2. Suppose that there is a $y \in Y_0$ such that $card(E \cap P^{-1}(y)) > 1$. By Lemma 2.5, there is a $(U_0, y) \in A(y) \equiv E \cap P^{-1}(y)$ such that $(U_0, y) \geq (U, y)$ for all $(U, y) \in A(U_0, y)$. Recall that $A(U_0, y) = \{(U, y) \in E | Z_+(U_0, U, y) = N_0(A(y))\}$, and

$$N_0(A(y)) = \min\{Z_+(U_1, U_2, y) | (U_1, y), (U_2, y) \in A(y), U_1 \neq U_2\}.$$

Take $(U_1^*, y)$, $(U_2^*, y) \in A(y)$, $U_1^* \neq U_2^*$, such that $Z_+(U_1^*, U_2^*, y) = N_0(A(y))$. By Lemma 2.1 and the invariance of $Y_0$, without loss of generality, we assume that $u(t, \cdot, U_1^*, y) - u(t, \cdot, U_2^*, y)$ has only simple zeros in $[0, 1]$ for $t \geq 0$, and $(U_1^*, y) > (U_2^*, y)$. By minimality of $E$ and Lemma 2.2, there are sequences $t_n \to \infty$ and $\{(U_n, y)\} \subset A(y)$ such that $\Pi_{t_n}(U_0, y) \to (U_2^*, y)$, $\Pi_{t_n}(U_n, y) \to (U_1, y)$ as $n \to \infty$. Similar to the arguments in Lemma 2.4 and Lemma 2.5, there is an $n_0 > 0$ such that $u(t_n, \cdot, U_0, y) - u(t_n, \cdot, U_n, y)$ has only simple zeros in $[0, 1]$, and $Z_+(U_0, U_n, y) = N_0(A(y))$ for $n \geq n_0$. Thus, $u(t, \cdot, U_0, y) - u(t, \cdot, U_n, y)$ has only simple zeros in $[0, 1]$, and $Z(u(t, \cdot, U_0, y) - u(t, \cdot, U_n, y)) = N_0(A(y))$ for $t \geq t_n, n \geq n_0$. This implies that $(1-\beta)(u(t, 0, U_0, y) - u(t, 0, U_n, y)) + \beta(u_x(t, 0, U_0, y) - u_x(t, 0, U, y))$ has constant sign for $t \geq t_n$, $n \geq n_0$. By choosing a larger $n_0$ if necessary, one has that $(1 - \beta)(u(t_n, 0, U_0, y) - u(t_n, 0, U_n, y)) + \beta(u_x(t_n, 0, U_0, y) - u_x(t_n, 0, U_n, y))$ has the same sign as $(1 - \beta)(U_2^*(0) - U_1^*(0)) + \beta(U_{2x}^*(0) - U_{1x}^*(0))$ for $n \geq n_0$. Since $(U_2^*, y) < (U_1^*, y)$, it follows from the above discussions that $(U_0, y) < (U_n, y)$ for $n \geq n_0$.

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Note that $\{(U_n, y)\} \subset A(U_0, y)$ and $(U_n, y) \leq (U_0, y)$ for $n \geq n_0$. We then have a contradiction. ■

**Definition 2.4.** *Given a flow $E$ with compact metric phase space $(E, d)$, points $x_1$, $x_2 \in E$ are said to be <u>proximal</u> if*

$$\inf_{t \in \mathbb{R}} d(x_1 \cdot t, x_2 \cdot t) = 0.$$

*A point $x \in E$ which is proximal only to itself is said to be <u>distal</u>. $E$ is called a <u>point-distal flow</u> if there is a distal point $x \in X$ such that its orbit $\{x \cdot t | t \in \mathbb{R}\}$ is dense in $E$. $E$ is <u>distal</u> flow if every point in $E$ is distal.*

*Let $p : E \to Y$ be a homomorphism of minimal flows. A point $x \in p^{-1}(y)$ is a <u>fibre distal point</u> if $x$ is not proximal with any point in $p^{-1}(y) \setminus \{x\}$. $E$ is a <u>fibre distal flow</u> if any point in $E$ is fibre distal.*

**Remark 2.1.** 1) A point-distal flow is necessarily minimal ([15], [40]). Distal flow laminates into minimal sets ([15]).

2) Let $E \subset X \times Y$ be a minimal invariant set of (1.3). By Theorem 2.6, if $y \in Y$ is such that $card E \cap P^{-1}(y) > 1$, then any two points $(U_1, y)$, $(U_2, y) \in E \cap P^{-1}(y)$ are proximal. To see this, let $(U_0, y_0)$ be a point such that $(U_0, y_0) = E \cap P^{-1}(y_0)$. Then there is a sequence $t_n \to \infty$ such that $y \cdot t_n \to y_0$ as $n \to \infty$. Take a subsequence if necessary, one sees that $\Pi_{t_n}(U_i, y) \to (U_0, y_0)$ $(i = 1, 2)$ as $n \to \infty$. Thus, a point $(U, y)$ in a minimal set $E$ of (1.3) is a fibre distal point if and only if $(U, y) = E \cap P^{-1}(y)$.

**Corollary 2.7.** *Let $E \subset X \times Y$ be a minimal set of (1.3). Then $E$ is point-distal if and only if $Y$ is point-distal.*

**Proof.** The 'only if' part is obvious since the projection $P : E \to Y$ is a flow homomorphism. We now prove the 'if' part. By Theorem 2.6, there is a residual subset $Y_0 \subset Y$ such that $card(E \cap P^{-1}(y)) = 1$ for $y \in Y_0$. Let $Y_1$ be the set of distal point of $Y$. $Y_1$ is a residual subset of $Y$ ([16]). Hence $Y^* = Y_0 \cap Y_1 \subset Y$ is also residual. Now let $y \in Y^*$ and $(U, y) = E \cap P^{-1}(y)$. If $(U^*, y^*) \in E$ is proximal to $(U, y)$, then $y^*, y$ are proximal. It follows that $y^* = y$. Since $card E \cap P^{-1}(y) = 1$, $(U^*, y^*) = (U, y)$. Above all, for any $y \in Y^*$, $(U, y) = E \cap P^{-1}(y)$ is a distal point of $E$. ■

**Corollary 2.8.** *Let $Y$ in (1.3) be a distal flow. Then any minimal set $E$ of (1.3) is point distal. Moreover, $(U, y) \in E$ is a distal point if and only if $(U, y) = E \cap P^{-1}(y)$.*

**Proof.** Note that in the case $Y$ is distal, a point in $E$ is a fibre distal point if and only if it is a distal point. The corollary then follows immediately from Corollary 2.7 and Remark 2.1 2). ■

We now discuss cases in which a minimal set $E$ of (1.3) can be a precise 1-1 extension of $Y$.

**Definition 2.5.** *A motion $\Pi_t(U_0, y_0)$ of (1.3) is said to be* <u>*uniformly stable*</u> *if for any $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$ such that if*

$$\|u(\tau, \cdot, U_0, y_0) - u(\tau, \cdot, U_1, y_0)\| < \delta(\epsilon)$$

*for some $(U_1, y_0) \in X \times Y$, and some $\tau \in \mathbb{R}^+$, then*

$$\|u(t + \tau, \cdot, U_0, y_0) - u(t + \tau, \cdot, U_1, y_0)\| < \epsilon$$

*for all $t \in \mathbb{R}^+$.*

**Theorem 2.9.** *A minimal set $E$ of (1.3) is a 1-1 extension of $Y$ provides one of the following conditions holds.*

*1) $E$ is fibre distal;*

*2) There is a $(U_0, y_0) \in E$ such that $\Pi_t(U_0, y_0)$ is uniformly stable;*

*3) $E$ is hyperbolic (that is, the linearized equation of the flow on $E$ admits an exponential dichotomy over $E$, see [38] for detail);*

*4) Equations $(1.1)_y$-(1.2) are monotone in the sense that $f_u(y, x, u, p) \leq 0$ for all $(y, x, u, p) \in Y \times [0, 1] \times \mathbb{R}^1 \times \mathbb{R}^1$.*

**Proof.** 1) follows from Remark 2.1 2).

2) By arguments in [33] or [35], the uniform stability implies the fibre distality.

3) This is proved in [38] in the case that $Y$ is an almost periodic minimal set. The proof in [38] can be carried over for general minimal set $Y$.

4) By applying the strong maximum principal ([19], [32]) for parabolic equations, one shows that the monotoness implies that each minimal set of (1.3) is fibre distal (see [37] for detail). ■

Remark 2.2. The statement 3) in Theorem 2.9 is in fact true for any $\omega$-limit set, that is, if an $\omega$-limit set $\omega(U_0, y_0)$ is hyperbolic, then it is a precise 1-1 extension of $Y$ (see [38]).

We end this section by stating a structure theorem of the $\omega$-limit set $\omega(U_0, y_0)$ for a positively bounded motion $\Pi_t(U_0, y_0) = (u(t, \cdot, U_0, y_0), y_0 \cdot t)$ of (1.3), that is, $u(t, \cdot, U_0, y_0)(t \geq 0)$ is bounded in $X$. Following [21] and the standard a priori estimates for parabolic equations, we know that if $u(t, \cdot, U_0, y_0)$ $(U_0 \in X)$ is bounded in $X$ for $t$ in the existence interval of the solution, then $u$ is a globally defined classical solution; moreover, for any $\delta > 0$, $\{u(t, x, U, g)|t \geq \delta\}$ is relatively compact both in $X$ and in $H^2(0, 1)$. Therefore $\omega(U_0, y_0)$ is a nonempty connected compact subset of $X \times Y$ ([20]).

**<u>Theorem 2.10.</u>** *Consider (1.3). Let $(U_0, y_0) \in X \times Y$ be such that the motion $\Pi_t(U_0, y_0)$ $(t > 0)$ is bounded. Then the $\omega$-limit set $\omega(U_0, y_0)$ contains at most two minimal sets. More precisely, one of the following is true:*

1) *$\omega(U_0, y_0) = E_1 \cup E_2 \cup E_{12}$, where $E_1$, $E_2$ are minimal sets, $E_{12} \neq \emptyset$. $E_{12}$ connects $E_1$, $E_2$ in the sense that if $(U_{12}, y) \in E_{12}$, then $\omega(U_{12}, y) \cap (E_1 \cup E_2) \neq \emptyset$, $\alpha(U_{12}, y) \cap (E_1 \cup E_2) \neq \emptyset$ (here $\alpha$ is referred to as the $\alpha$-limit set).*

2) *$\omega(U_0, y_0) = E_1 \cup E_{11}$, where $E_1$ is minimal, $E_{11} \neq \emptyset$, $E_{11}$ connects $E_1$ in the sense that if $(U_{11}, y) \in E_{11}$, then $\omega(U_{11}, y) \cap E_1 \neq \emptyset$, $\alpha(U_{11}, y) \cap E_1 \neq \emptyset$.*

3) *$\omega(U_0, y_0)$ is a minimal invariant set.*

**<u>Theorem 2.11.</u>** *Any two minimal invariant sets $E_1$, $E_2$ of (1.3) are separated in the following sense:*

1) *$[a_1(y), b_1(y)] \cap [a_2(y), b_2(y)] = \emptyset$ for all $y \in Y$, where*

$$a_i(y) = \min\{(1 - \beta)U(0) + \beta U_x(0)|(U, y) \in E_i \cap P^{-1}(y)\},$$
$$b_i(y) = \max\{(1 - \beta)U(0) + \beta U_x(0)|(U, y) \in E_i \cap P^{-1}(y)\}$$

*($i = 1, 2$).*

2) *Without loss of generality, assume that $a_1(y^*) - b_2(y^*) > 0$ for some $y^* \in Y$. Then there is a $\delta > 0$ such that $a_1(y) - b_2(y) \geq \delta$ for all $y \in Y$.*

The proof of Theorem 2.10 and Theorem 2.11 are almost identical to those in the almost periodic case ([37]).

**<u>Remark 2.3.</u>** There is an example in the almost periodic case in which an $\omega$-limit set contains precisely two minimal sets (see [37]).

## 3. Almost Automorphic and Almost Periodic Extensions

In this section, we give our particular attentions to cases in which flow $Y$ in (1.3) is either an almost automorphic or an almost periodic minimal flow.

**Definition 3.1.** *1) Let $E$ be a complete metric space. A continuous function $f : \mathbb{R}^1 \to E$ is said to be <u>almost periodic</u> if from every sequence $\{t'_n\}$ one can extract a subsequence $\{t_n\} \subset \mathbb{R}^1$ such that $\lim_{n \to \infty} f(t + t_n)$ exists uniformly in $t \in \mathbb{R}^1$. $f$ is said to be <u>almost automorphic</u> if $\{t_n\} \subset \mathbb{R}^1$ and $g : \mathbb{R}^1 \to E$ are such that $f(t_n + t) \to g(t)$ as $n \to \infty$ pointwise in $t \in \mathbb{R}^1$, then $g(t - t_n) \to f(t)$ as $n \to \infty$ pointwise.*

*2) Let $E$ be a flow with compact metric phase space $E$. A point $x \in E$ is called an <u>almost periodic (almost automorphic) point</u> if its orbit $x \cdot t$ is an almost periodic (almost automorphic) function of $t$. The flow $E$ is called <u>almost periodic</u> (almost automorphic) <u>minimal</u> if it contains an almost periodic (almost automorphic) point $x_0 \in E$ such that the orbit $\{x_0 \cdot t\}$ is dense in $E$.*

**Remark 3.1.** 1) By definition, an almost periodic point (function) is necessarily an almost automorphic point (function) ([18], [26]). We shall see in Lemma 3.1 that an almost automorphic point $x_0$ is necessarily a distal point in a dynamical system. It then follows from Remark 2.1 that $cl\{x_0 \cdot t\}$ must be minimal.

2) It is well known that the flow is almost periodic minimal if and only if it is equicontinuous ([26], [35]). Moreover, any almost periodic minimal flow is distal ([15]).

**Definition 3.2.** *Consider a homomorphism of minimal flows $p : E \to Y$ with $Y$ being almost periodic minimal. $E$ is an <u>(almost automorphic)</u> <u>almost periodic</u> <u>extension</u> of $Y$ if $E$ is an (almost) 1-1 extension of $Y$.*

**Remark 3.2.** Let $E$, $Y$, $p$ be as above.

1) If $E$ is an almost automorphic extension of $Y$, then $E$ is an almost automorphic minimal set since it is clear that the set $\{x \in p^{-1}(y) | card\, p^{-1}(y) = 1, y \in Y\}$ consists of almost automorphic points.

2) If $E$ is an almost periodic extension of $Y$, then $E$ is an almost periodic minimal set since $p$ in this case is an isomorphism, and $E = p^{-1}(Y)$ is equicontinuous (Remark 3.1 2)).

**Lemma 3.1.** *Let $p : E \to Y$ be a homomorphism of minimal flows and assume that $E$ is a compact metric space, $Y$ is almost periodic minimal. If $E$ is an almost automorphic extension of $Y$, then $\{x \in E | x$ is an almost automorphic point $\} = \{x \in E | x$ is a distal point $\} = \{x \in E | card\, p^{-1}(px) = 1\} = \{x \in p^{-1}(y_0) | q : y \to p^{-1}(y)$ is continuous at $y_0 \in Y$ with the Hausdroff metric on $2^E\}$.*

**Proof.** Let $Y_0 \subset Y$ be the residual set satisfying properties of Lemma 2.2, i.e. $Y_0 = \{y \in Y | q : Y \to 2^E, y \mapsto p^{-1}(y)$ is continuous at $y_0$ in the Housdroff metric of $2^E\}$. Since $E$ is an almost automorphic extension of $Y$, by Lemma 2.2, it is easy to see that $Y_0 = \{y \in Y | card p^{-1}(y) = 1\}$. Adopting arguments from Corollary 2.7 and Corollary 2.8, one has that $Y_0 = \{y \in Y | x = p^{-1}(y)$ is distal $\}$. Let $Y^* = \{y \in Y |$ there is an almost automorphic point $x \in p^{-1}(y)\}$. Then $Y_0 \subset Y^*$ (Remark 3.2 1)). Now, if $y \in Y$ is such that there is an almost automorphic point $x \in p^{-1}(y)$, then we claim that $card p^{-1}(y) = 1$. If not, then there is a $x^* \in p^{-1}(y)$ such that $x^* \neq x$. Fix a $y_0 \in Y_0$ and denote $x_0 = p^{-1}(y_0)$. Let $\{t_n\}$ be a sequence such that $x_0 \cdot t_n \to x^*$ as $n \to \infty$. Since $x \cdot (-t_n) \to x_0$, it follows that $x_0 \cdot t_n \to x$ as $n \to \infty$, that is, $x^* = x$. This is a contradiction. ∎

**Corollary 3.2.** *1) $Y$ is almost automorphic minimal if and only if it contains residually many almost automorphic points.*

*2) $Y$ is almost periodic minimal if and only if it consists of almost automorphic points.*

Proof. 1) By [39], $Y$ is almost automorphic minimal if and only if it is an almost automorphic extension of an almost periodic minimal flow $\bar{Y}$. Note that an almost automorphic minimal flow is point distal and a point distal flow consists of residually many distal points ([16]). 1) then follows from Lemma 3.1.

2) The proof is a straightforward application of Lemma 3.1 and the above part 1). We only note that an almost periodic minimal flow is distal. A more general situation is discussed in ([39]). ∎

**Theorem 3.3.** *Let $E \subset X \times Y$ be a minimal set of (1.3). Then $E$ is almost automorphic if and only if $Y$ is almost automorphic.*

**Proof.** If $E$ is almost automorphic, then $Y$ is clearly almost automorphic. We now assume that $Y$ is almost automorphic. By Theorem 2.6 and Corollary 3.2, the set $Y^* = \{y \in Y | y$ is almost automorphic, $card E \cap P^{-1}(y) = 1\}$ is a residual subset of $Y$. Now take $y \in Y^*$ and denote $(U, y) = E \cap P^{-1}(y)$. Let $\{t_n\}$ be a sequence such that $\Pi_{t_n}(U, y)$ converges to some $(U^*, y^*) \in E$ as $n \to \infty$. Since $y^* \cdot (-t_n) \to y$ as $n \to \infty$ and $card E \cap P^{-1}(y) = 1$, it follows that $\Pi_{-t_n}(U^*, y^*) \to (U, y)$ as $n \to \infty$ (take subsequence if necessary). This shows that $(U, y)$ is an almost automorphic point. ∎

**Theorem 3.4.** *Let $Y$ in (1.3) be almost periodic minimal. Then the following holds.*

*1) Any minimal set $E \subset X \times Y$ of (1.3) is an almost automorphic extension of $Y$.*

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*2) A minimal set $E \subset X \times Y$ of (1.3) is an almost periodic extension of $Y$ provided that one of the conditions in Theorem 2.9 holds.*

*3) If $E \subset X \times Y$ of (1.3) is almost periodic minimal, the it is an almost periodic extension of $Y$.*

**Proof.** 1) is a corollary of Theorem 2.6.

2) follows from Theorem 2.9.

3) is a consequence of Lemma 3.1, Corollary 3.2, and 1) of the current theorem. ∎

**Remark 3.3.** Theorem 3.3, 3.4 are 'sharp' in the following sense.

1) Theorem 3.3 implies that dynamics of (1.3) is 'closed' in the category of almost automorphy. Nevertheless, as shown in Section 5 (see also [24], [25], [37]), dynamics of (1.3) is not 'closed' in the category of almost periodicity, that is, even $Y$ is almost periodic minimal, a minimal set $E$ of (1.3) can be non-almost periodic almost automorphic.

2) Dynamics of (1.3) is however 'closed' for periodicity since any almost 1-1 extension of a periodic minimal set $Y$ is a precise 1-1 extension of $Y$ by the periodicity. In fact, in this case, it is shown in [5], [9] that any $\omega$-limit set of (1.3) is a precise 1-1 extension of $Y$, that is, a periodic minimal set with the same period as that of $Y$.

**Remark 3.4.** Consider a family of scalar ODEs:

$$u' = f(y \cdot t, u), \quad u \in \mathbb{R}^1, \quad y \in Y, \tag{3.1}$_y$$

where $Y$ is a compact minimal flow. Equations $(3.1)_y$ generates a skew-product flow $\Pi_t$ on $\mathbb{R}^1 \times Y$,

$$\Pi_t(u_0, y) = (u(t, u_0, y), y \cdot t), \tag{3.2}$$

where $u(t, u_0, y)$ is the solution of $(3.1)_y$ with $u(0, u_0, y) = u_0$.

Let the function $f$ in $(1.1)_y$ be independent of $x$, $u_x$, and let $\beta = 0$ in (1.2). Then (3.2) defines a subflow of (1.3). Thus, all our results so far hold true for (3.2) with $X = \mathbb{R}^1$.

**4. Applications**

Let us consider a scalar parabolic equation of following form

$$\begin{cases} u_t = u_{xx} + F(t, x, u, u_x), & t > 0, \quad 0 < x < 1, \\ \beta u(t, 0) + (1 - \beta)u_x(t, 0) = 0, \beta u(t, 1) + (1 - \beta)u_x(t, 1) = 0, t > 0, \end{cases} \tag{4.1}$$

where $\beta = 0$ or 1, $F$ is $C^2$, and for every compact set $K \subset [0,1] \times \mathbb{R}^1 \times \mathbb{R}^1$, $F(t, x, u, p)$ with all its partial derivatives up to order 2 are uniformly continuous and bounded on $\mathbb{R}^1 \times K$. We shall show that (4.1) with all its 'translated' and 'limiting' equations together generate a skew-product semi-flow of form $(1.1)_y$-$(1.2)$, or $(1.3)$. The construction is based on ideas of classical topological dynamical systems (see, for example, [35]).

**Standard Hypothesis:** Consider a $C^2$ function $F : \mathbb{R}^1 \times [0,1] \times \mathbb{R}^m \to \mathbb{R}^n$. $F$ is said to be standard if for any compact subset $K \subset [0,1] \times \mathbb{R}^m$, $F$ and all its partial derivatives up to order 2 are uniformly continuous and bounded on $\mathbb{R}^1 \times K$.

**Proposition 4.1.** ([22], [23], [45]) Consider a function $F$ which satisfies the standard hypothesis. Then there exists a Lipschitz flow $Y_0$ (mean $(y, t) = y \cdot t$ is Lipschitz in both $y$ and $t$) with $Y_0 = H(F)$ (the hull of $f$) being a compact metric space, and a function $f : Y_0 \times [0,1] \times \mathbb{R}^m \to \mathbb{R}^n$ such that the following holds:

1) $f : Y_0 \times [0,1] \times \mathbb{R}^m \to \mathbb{R}^n$, $(y, x, z) \mapsto f(y, x, z)$ is $C^2$ in $x, z$ and Lipschitz in $y$;

2) There is a $y_0 \in Y_0$ such that

$$f(y_0 \cdot t, x, z) \equiv F(t, x, z).$$

**Proof.** Let $C$ be the space of continuous functions $S : \mathbb{R}^1 \times [0,1] \times \mathbb{R}^m \to \mathbb{R}^n$ with the compact open topology. This topology is metrizable. To be more precise, for $S_1, S_2 \in C$, define a metric $d$ as follows,

$$d(S_1, S_2) = \sum_{n=1}^{\infty} \frac{\min\{1, \|S_1 - S_2\|_n\}}{2^n},$$

where $\|S_1 - S_2\|_n = \sup_{|t| \leq n, 0 \leq x \leq 1, |z| \leq n} |S_1(t, x, z) - S_2(t, x, z)|$. It is easy to see that the translation $(S, t) \mapsto S_t$, $S_t(s, x, z) = S(s + t, x, z)$ defines a flow on $(C, d)$ ([35]) and $Y_0 = H(F) = cl\{F_t | t \in \mathbb{R}^1\}$, the hull of $F$, is compact (Ascoli's Theorem), and translation invariant.

Define $f : Y_0 \times [0,1] \times \mathbb{R}^m \to \mathbb{R}^n$, $(y, x, z) \mapsto y(0, x, z)$ and let $y_0 = F$. Then $f(y_0 \cdot, x, z) = y_0 \cdot t(0, x, z) = y_0(t, x, z) = F(t, x, z)$. For other properties of $Y_0$ and $f$, see [22], [23], [45] for more details. ∎

We note that $f$ needs only to be Lipschitz in $t$ for the above proposition to be held.

**Proposition 4.2.** *Let $F : \mathbb{R}^1 \times [0,1] \times \mathbb{R}^m \to \mathbb{R}^n$ be a function satisfying the standard hypothesis and let $Y_0 = H(F)$ be as in Lemma 4.1. Then the following holds.*

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*1) If for each $(x, z) \in [0,1] \times \mathbb{R}^m$, $F(\cdot, x, z) : \mathbb{R}^1 \to \mathbb{R}^m$ is an almost automorphic function, then $Y_0$ is an almost automorphic minimal flow.*

*2) If for each $(x, z) \in [0,1] \times \mathbb{R}^m$, $F(\cdot, x, z) : \mathbb{R}^1 \to \mathbb{R}^m$ is an almost periodic function, then $Y_0$ is an almost periodic minimal flow.*

**Proof.** 1) We only observe that $y_0 = F$ is an almost automorphic point of $Y_0$.

2) is a standard fact (see [35]). ■

Now, consider (4.1) and let $f$ be the function corresponding to $F$ in Proposition 4.1. Then (4.1) gives rise to a family of equations over $Y_0 = H(F)$:

$$\begin{cases} u_t = u_{xx} + f(y \cdot t, x, u, u_x), \quad t > 0, \quad 0 < x < 1, \\ \beta u(t, 0) + (1 - \beta) u_x(t, 0) = 0, \beta u(t, 1) + (1 - \beta) u_x(t, 1) = 0, t > 0, \end{cases} \tag{4.2}_y$$

where $\beta = 0$ or $1$, $y \in Y_0$. Note that $f(y \cdot t, x, u, p) = y(t, x, u, p)$ and if $y_0 = F$, then $(4,2)_{y_0}$ coincides with (4.1). Equations $(4.2)_y$ $(y \in Y_0)$ then generates a (local) skew-product semi-flow $\Pi_t$ on $X \times Y_0$ (here $X$ is as in (1.3)):

$$\Pi_t(u_0, y) = (u(t, \cdot, u_0, y), y \cdot t). \tag{4.3}$$

Since $Y_0$ is compact invariant, it contains a minimal set $Y$. All results in section 2 therefore apply to the (local) skew-product semi-flow $\Pi_t$ restricted to $X \times Y$.

*4.1. Asymptotic behavior of positively bounded solutions*

We now discuss asymptotic behavior of positively bounded solutions of (4.1).

**Definition 4.1.** *Consider a positively bounded solution $u(t, x)$ of (4.1), that is, $\{u(t, \cdot)|t \geq 0\}$ is bounded in $X$. $u(t, \cdot)$ is said to be <u>asymptotically almost periodic</u> if there is an almost periodic function $u^* : \mathbb{R}^1 \to X$, $t \mapsto u^*(t, \cdot)$, such that $\|u(t, \cdot) - u^*(t, \cdot)\| \to 0$ as $t \to \infty$.*

**Proposition 4.3.** *Consider (4.1) and assume that the function $F$ satisfies the condition 2) of Proposition 4.2. Let $u(t, \cdot)$ be a positively bounded solution of (4.1). Then $u(t, \cdot)$ is asymptotically almost periodic if and only if the $\omega$-limit set $\omega(U_0, F)$ in (4.3) is an almost periodic extension of $H(F)$, here $U_0(\cdot) = u(0, \cdot)$.*

**Proof.** Suppose that $u(t, \cdot)$ is asymptotically almost periodic, that is, $\|u(t, \cdot) - u^*(t, \cdot)\| \to 0$ as $t \to \infty$ for some almost periodic function $u^* : \mathbb{R}^1 \to X$, $t \mapsto u^*(t, \cdot)$. Let $t_n \to \infty$ be such that $\Pi_{t_n}(U_0, F) \to (\tilde{U}, F) \in \omega(U_0, F)$ and $u^*(t + t_n, \cdot) \to u^*(t, \cdot)$ $(t \in \mathbb{R}^1)$ as $n \to \infty$.

Then $\|u(t,\cdot,\tilde{U},F) - u^*(t,\cdot)\| = \lim_{n\to\infty} \|u(t+t_n,\cdot) - u^*(t+t_n,\cdot)\| = 0$. Hence $u^*(t,\cdot)$ is an almost periodic solution of (4.1) (or $(4.2)_F$), and $\omega(U_0,F) = cl\{\Pi_t(U^*,F)|t\in\mathbb{R}^1\}$, where $U^* = u^*(0,\cdot)$. Therefore, $\omega(U_0,F)$ is almost periodic minimal. By Theorem 3.4 3), $\omega(U_0,F)$ is an almost periodic extension of $H(F)$.

Conversely, suppose that $\omega(U_0,F)$ is an almost periodic extension of $H(F)$. Let $(U^*,F) = \omega(U_0,F)\cap P^{-1}(F)$. Then $u^*(t,\cdot,U^*,F)$ is an almost periodic solution of $(4.2)_F$. Now, one must have $\|u(t,\cdot) - u^*(t,\cdot,U^*,F)\| \to 0$ as $t\to\infty$ (hence $u(t,\cdot)$ is asymptotically almost periodic). For otherwise, there is a sequence $t_n \to \infty$ such that $\Pi_{t_n}(U_0,F) \to (\tilde{U}_0,\tilde{F})$ and $\Pi_{t_n}(U^*,F) \to (\tilde{U}^*,\tilde{F})$ as $n \to \infty$, where $\tilde{U}_0 \neq \tilde{U}^*$. Obviously, $(\tilde{U}_0,\tilde{F})$, $(\tilde{U}^*,\tilde{F}) \in \omega(U_0,F)$. This implies that $\omega(U_0,F)$ is not a 1-1 extension of $H(F)$ or $\omega(U_0,F)$ is not an almost periodic extension of $H(F)$. This is a contradiction. $\blacksquare$

For a positively bounded solution $u(t,x)$ of (4.1), denote $U_0(\cdot) = u(0,\cdot)$. By Proposition 4.3, $u(t,\cdot)$ is asymptotically almost periodic if and only if $\omega(U_0,F)$ is minimal and consists of almost automorphic points. Now let us assume in the moment that $\omega(U_0,F)$ is minimal but not almost periodic. By Theorem 3.4, $\omega(U_0,F)$ contains only residually many almost automorphic points, and each almost automorphic point $(U^*,y^*)$ corresponds to an almost automorphic solution $u(t,x,U^*,y^*)$ of $(4.2)_{y^*}$. Comparing with the asymptotic almost periodicity, one then conclude that $u(t,\cdot)$ is 'almost' asymptotically almost automorphic. Note that even $\omega(U_0,F)\cap P^{-1}(F)$ is a single almost automorphic point, $u(t,x)$ may not be asymptotically almost automorphic in general.

*4.2. Asymptotically almost automorphic or asymptotically almost periodic equations*

Let $F$ in (4.1) be either asymptotically almost automorphic or asymptotically almost periodic, that is, $F = F_0 + h$, where $F_0$ satisfies condition 1) or 2) in Proposition 4.2, $h \to 0$ as $t \to \infty$ uniformly on compact set $K \subset [0,1] \times \mathbb{R}^2$.

In this case, if $u_0 \in X$ is such that the solution $u(t,\cdot,u_0,F)$ of (4.1) is bounded in $X$ for $t > 0$, then the $\omega$-limit set $\omega(u_0,F) \subset X \times Y$ is compact invariant under flow $\Pi_t : X \times Y \to X \times Y$, where $Y = H(F_0)$ is either an almost automorphic or an almost periodic minimal set. By Theorem 3.3, $\omega(u_0,F)$ contains an almost automorphic minimal set $E$. Let $(u^*,y^*)$ be an almost automorphic point in $E$. Then there is a sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ such that $u(t+t_n,\cdot,u_0,F) \to u(t,\cdot,u^*,y^*)$ as $n \to \infty$. Note that $u(t,x,u^*,y^*)$ is an almost automorphic solution of $(4.2)_{y^*}$ with $u(0,x,u^*,y^*) = u^*(x)$.

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If $\omega(u_0, F)$ is minimal and $(u^*, F_0) \in \omega(u_0, F)$ is an almost automorphic point, then for 'almost all' sequences $t_n \to \infty$, $u(t_n, \cdot, u_0, F) - u(t_n, \cdot, u^*, F_0) \to 0$ as $n \to \infty$. Roughly speaking, $u(t, \cdot, u_0, F)$ is 'almost' asymptotically almost automorphic. If $Y = H(F_0)$ is almost periodic minimal, similar to Proposition 4.3, we have

**Proposition 4.4.** *Suppose that $Y = H(F_0)$ is almost periodic minimal and $u(t, \cdot, u_0, F)$ is a positively bounded solution of (4.1). Then $u(t, \cdot, u_0, F)$ is asymptotically almost periodic if and only if $\omega(u_0, F)$ is an almost periodic extension of $H(F_0)$.*

**Proof.** We use similar arguments as in Proposition 4.3. In fact, if $u(t, \cdot, u_0, F)$ is asymptotically almost periodic, that is, $u(t, \cdot, u_0, F) - u^*(t, \cdot) \to 0$ as $t \to \infty$ for some almost periodic function $u^*(t, \cdot)$, then we can show that $u^*(t, \cdot)$ is an almost periodic solution of $(4.2)_{F_0}$ and $\omega(u_0, F) = cl\{\Pi_t(U^*, F_0) | t \in \mathbb{R}^1\}$, where $U^*(\cdot) = u^*(0, \cdot)$. Hence $\omega(u_0, F)$ is an almost periodic extension of $H(F_0)$. Conversely, if $\omega(u_0, F)$ is an almost periodic extension of $H(F_0)$, let $(u^*, F_0) = \omega(u_0, F) \cap P^{-1}(F_0)$. Then $u(t, \cdot, u^*, F_0)$ is an almost periodic solution of $(4.2)_{F_0}$ and $\|u(t, \cdot, u_0, F) - u(t, \cdot, u^*, F_0)\| \to 0$ as $t \to \infty$. $\blacksquare$

*4.3. System of recursively coupled scalar equations*

We now look at a system of recursively coupled scalar equations of the following form:

$$\begin{cases} u_t^k = u_{xx}^k + F^k(t, x, u^k, u_x^k, u^{k-1}, u_x^{k-1}, \cdots, u^1, u_x^1), & t > 0, \quad 0 < x < 1, \\ \beta^k u^k(t, 0) + (1 - \beta^k)u_x^k(t, 0) = 0, \beta^k u^k(t, 1) + (1 - \beta^k)u_x^k(t, 1) = 0, t > 0, \end{cases} \quad (4.4)_k$$

where $\beta^k = 0$ or $1$, $k = 1, 2, \cdots, n$, $F^k(1 \leq k \leq n)$ satisfies the condition 2) of Proposition 4.2. By Proposition 4.1 and Proposition 4.2, there are functions $f^k(1 \leq k \leq n)$ and an almost periodic minimal flow $Y$ such that $f^k(y_0 \cdot t, x, u^k, p^k, u^{k-1}, p^{k-1}, \cdot, u^1, p^1) \equiv F^k(t, x, u^k, p^k, u^{k-1}, p^{k-1}, \cdot, u^1, p^1)$ $(1 \leq k \leq n)$, here $y_0 = (F^1, F^2, \cdots, F^n)^\top$ and $Y = H(y_0)$. Then $(4.4)_k$ $(1 \leq k \leq n)$ gives rise to equations over $Y$:

$$\begin{cases} u_t^k = u_{xx}^k + f^k(y \cdot t, x, u^k, u_x^k, u^{k-1}, u_x^{k-1}, \cdots, u^1, u_x^1), & t > 0, \quad 0 < x < 1, \\ \beta^k u^k(t, 0) + (1 - \beta^k)u_x^k(t, 0) = 0, \beta^k u^k(t, 1) + (1 - \beta^k)u_x^k(t, 1) = 0, t > 0, \end{cases} \quad (4.5)_{k,y}$$

where $1 \leq k \leq n$, $y \in Y$. For each $k$ with $1 \leq k \leq n$, choose a fractional power space $X^k$ as in section 1. Given $k_0$ with $1 \leq k_0 \leq n$, system $(4.5)_{k,y}$, $1 \leq k \leq k_0$, $y \in Y$ then generates a (local) skew-product semi-flow $\Pi_t^{k_0}$ on $X(k_0) \times Y$, where $X(k_0) = \Pi_{k=1}^{k_0} X^k$,

$$\Pi_t^{k_0}(u_0^{k_0}, u_0^{k_0-1}, \cdots, u_0^1, y)$$

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$$= (u^{k_0}(t, \cdot, u_0^{k_0}, \cdots, u_0^1, y), \cdots, u^2(t, \cdot, u_0^2, u_0^1, y), u^1(t, \cdot, u_0^1, y)), y \cdot t), \qquad (4.6)_{k_0}$$

where $u^k(t, x, u_0^k, \cdots, u_0^1, y)$ is the solution of $(4.5)_{k,y}$ with $u^k(0, \cdot, u_0^k, \cdots, u_0^1, y) = u_0^k(\cdot)$ $(1 \le k \le k_0)$.

Observe that for each $1 < k_0 \le n$, $\Pi_t^{k_0}$ is also a (local) skew-product semi-flow on $X^{k_0} \times (X(k_0 - 1) \times Y)$, that is,

$$\Pi_t^{k_0}(u_0^{k_0}, U_0^{k_0-1}, y) = (u^{k_0}(t, \cdot, y, u_0^{k_0}, U_0^{k_0-1}, y), \Pi_t^{k_0-1}(U_0^{k_0-1}, y)), \qquad (4.7)_{k_0}$$

where $u_0^{k_0} \in X^{k_0}$, $U_0^{k_0-1} = (u_0^{k_0-1}, \cdots, u_0^1)^\top \in X(k_0 - 1)$. Let $X^* = X(n)$ and $\Pi_t = \Pi_t^n$. Then $(4.6)_n$ or $(4.7)_n$ is the (local) skew-product semi-flow on $X^* \times Y$ generated by $(4.5)_{k,y}$, $1 \le k \le n$, $y \in Y$.

**Proposition 4.5.** *Consider the (local) skew-product semi-flow $\Pi_t$ on $X^* \times Y$. The following holds.*

*1) Any minimal set $X \subset X^* \times Y$ of $\Pi_t$ is an almost automorphic extension of $Y$.*

*2) If $F_{u^k}^k \le 0$ for $k = 1, 2 \cdots, n$, then any minimal set $E \subset X^* \times Y$ of $\Pi_t$ is an almost periodic extension of $Y$.*

**Proof.** We prove the Theorem by induction. For $k = 1$, the Theorem follows from Theorem 3.4.

Suppose that the Theorem holds for some $k_0$ $(1 \le k_0 < n)$. Consider $(4.7)_{k_0+1}$. Let $E^{k_0+1} \subset X(k_0+1) \times Y$ be a minimal set of $\Pi_t^{k_0+1}$ and let $P^{k_0+1} : X(k_0+1) \times Y \to X(k_0) \times Y$ be the natural projection. Then $P^{k_0+1}(E^{k_0+1}) \equiv E^{k_0} \subset X(k_0) \times Y$ is a minimal set of $\Pi_t^{k_0}$. Restricting $\Pi_t^{k_0+1}$ to $X^{k_0+1} \times E^{k_0}$ and applying Theorem 2.6, one has that $E^{k_0+1}$ is an almost 1-1 extension of $E^{k_0}$. Since an almost 1-1 extension of an almost automorphic extension of $Y$ is also an almost automorphic extension of $Y$ (see arguments in Theorem 3.3), 1) is proved.

To prove 2), note that if the condition 2) is satisfied, then, by Theorem 2.9, $E^{k_0+1}$ is a precisely 1-1 extension of $E^{k_0}$ which is an almost periodic extension of $Y$. ∎

## 5. Examples

In this section, we describe two examples from [18] and [24] in which a scalar almost periodic ODE admits a non-almost periodic almost automorphic solution.

The idea is to construct certain almost periodic scalar ODE:

$$x' = F(t, x) \qquad (5.1)$$

which admits no almost periodic solution but a bounded solution. By Lemma 4.1, (5.1) generates a family of almost periodic scalar equations on $Y = H(F)$:

$$x' = f(y \cdot t, x), \tag{5.2}_y$$

where $f : Y \times \mathbb{R}^1 \to \mathbb{R}^1$ is a function such that $f(y_0 \cdot t, x) \equiv F(t, x)$ for $y_0 = F$. If (5.1) has no almost periodic solution, then the skew-product flow $\Pi_t : \mathbb{R}^1 \times Y \to \mathbb{R}^1 \times Y$,

$$\Pi_t(x_0, y) = (x(t, x_0, y), y \cdot t) \tag{5.3}$$

admits no almost periodic point. Let $x(t, x_0)$ be a bounded solution of (5.1) and consider $\omega(x_0, y_0)$. By Remark 3.4 and Theorem 3.3, there is an almost automorphic point $(x^*, y^*)$ in $\omega(x_0, y_0)$. Now, $x(t, x^*)$ is a non-almost periodic almost automorphic solution to the almost periodic scalar equation $(5.2)_{y^*}$.

**Example 1** (Johnson [24]). A linear almost periodic scalar ordinary differential equation of form

$$x' + A(t)x = B(t) \tag{5.4}$$

is constructed in [24] as follows.

By using so called Conley-Miller functions, sequences $A_n(t)$, $B_n(t)$ of periodic functions can be constructed with the following properties:

1) $A_n(t)$, $B_n(t)$ are $2^n$-periodic $(n = 1, 2, \cdots)$ and are uniformly convergent. Let $A(t) = \lim_{n \to \infty} A_n(t)$, $B(t) = \lim_{n \to \infty} B_n(t)$. Then $A(t)$, $B(t)$ are almost periodic, $\int_0^t A(s)ds \to \infty$ as $t \to \infty$ and $mod B \subset mod A$.

2) For each $n$, the solution $x_n(t)$ of the scalar equation

$$x' + A_n(t)x = B_n(t) \tag{5.5}$$

with initial value 0 is $2^n$-periodic,

$$x_n(2^n) = \begin{cases} \frac{1}{5}, & n \geq 4, \quad n \quad odd \\ 0, & n \quad even \end{cases}$$

and $|x_n(t)| \leq 1$.

It is clear that $x_0(t) = \lim_{n \to \infty} x_n(t)$ is a bounded solution of (5.4). Now, if (5.4) has an almost periodic solution $\bar{x}(t)$, then $mod\bar{x} \subset modA$ ([18]). Since $\lim_{n \to \infty} A(2^n)$ exists, $\lim_{n \to \infty} \bar{x}(2^n)$ exists ([18]). Let $y(t) = \bar{x}(t) - x_0(t)$. Then $y(t)$ satisfies

$$y' + A(t)y = 0. \tag{5.6}$$

It follows that $y(t) = \bar{x}(0)e^{-\int_0^t A(s)ds} \to 0$ as $t \to \infty$. In particular, $\lim_{k \to \infty} \bar{x}(2^{2k}) = \lim_{k \to \infty} x_0(2^{2k}) = 0$ and $\lim_{k \to \infty} \bar{x}(2^{2k+1}) = \lim_{k \to \infty} x_0(2^{2k+1}) = \frac{1}{5}$. Thus, $\lim_{n \to \infty} \bar{x}(2^n)$ does not exist, a contradiction.

**Example 2** (Fink [18]). Consider a differential equation on the torus

$$x' = f(t, x) \tag{5.7}$$

where $f(t+1, x) = f(t, x+1) = f(t, x)$. Let $x(t, \eta)$ be the solution of (5.7) with $x(0, \eta) = \eta$. Consider the Poincaré map $\psi$, $\eta \mapsto x(1, \eta)$. It is well known that when the rotation number $\rho$ of $\psi$ is irrational, the limit set $S'$ of $\{\psi^n(\eta) \bmod 1, n = 1, 2, \cdots\}$ is either $[0, 1]$ or is a cantor set (see [18]).

Now, let $f$ in (5.7) be such that $S'$ is a cantor set. Consider

$$x' = f(x + \rho t, t) - \rho. \tag{5.8}$$

Clearly, (5.8) is quasi-periodic dependent on $t$ with frequences 1 and $\rho$. It is shown by Fink ([18], example 12.5) that equation (5.8) admits a bounded solution but admits no almost periodic solution.

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