# Homework 1

① Suppose a scheme defined on an interval $[a,b]$ with Dirichlet boundary condition is written as

$$\sum_{j=-k_1}^{k_1} a_j U_{i+j}^{n+1} = \sum_{j=-k_2}^{k_2} b_j U_{i+j}^{n}$$

where (a) $a_0 > 0$, $a_j \leq 0$ if $j \neq 0$; and

(b) $b_j \geq 0$ for all $j$; and

(c) $\sum_{j=-k_1}^{k_1} a_j = \sum_{j=-k_2}^{k_2} b_j$.

Show that the scheme is stable in $l^\infty$.

② Use the MUSCL scheme to solve the following problems.

(a) $$\begin{cases} u_t + u_x = 0, & t>0, x\in(0,1) \\ u(x,0)=1, \ x\in[\frac{1}{3},\frac{2}{3}]; & 0, \ x\in[0,\frac{1}{3})\cup(\frac{2}{3},1] \\ u(0,t) = u(1,t), & t>0 \end{cases}$$

Compute the solution at $T=10$, using 100 cells in $(0,1)$. Plot the numerical solution against the exact solution at $T$.

(b) $$\begin{cases} u_t + (\frac{1}{2}u^2)_x = 0, & t>0, \ x\in\mathbb{R} \\ u(x,0) = 1+\sin(2\pi x), & x\in\mathbb{R} \end{cases}$$

Compute the solution at $T=1$, using 100 cells in $(0,1)$. Note that the solution is periodic in $x$ just as in (a).

③ Consider the equation $u_t + a u_x = 0$, where "a" is a constant. Under what CFL number is the magnitude of the Fourier symbol of the Lax-Friedrichs scheme for the above equation bounded by 2? This implies that BFECC applied to the Lax-Friedrichs scheme is 2nd order and stable in $l^2$ under the CFL number.

④ Suppose you solve the Poisson equation

$$\begin{cases} -\Delta u = f, & \text{in } \Omega = [0,1]\times[0,1] \\ u|_{\partial\Omega} = g \end{cases}$$

on a rectangular grid with the following 5-point scheme (with $\Delta x = \Delta y = h$)

$$-\left\{ \frac{U_{i+1,j} - 2U_{ij} + U_{i-1,j}}{h^2} + \frac{U_{i,j+1} - 2U_{ij} + U_{i,j-1}}{h^2} \right\} = f_{ij}.$$

Design a multi-grid method so that you can solve the above scheme with optimal complexity. Write a pseudo code for your method.