MATH1502 A2B2, Test 1, June 7, 2007

NAME:

Be sure to show your work. Answers without explanations are not acceptable.

1. (10 points) Calculate the limit $\lim_{x\to\infty}(\cos(1/x))^x$.

$$\lim (\cos 1/x)^x$$
$$= \lim e^{x \ln(\cos 1/x)}$$

Since $\lim\limits_{x\to\infty} x \ln(\cos 1/x) = \lim\limits_{x\to\infty} \frac{\ln(\cos 1/x)}{1/x}$

$$= \lim_{x\to\infty} \frac{\sin\frac{1}{x}\cdot\left(-\frac{1}{x^2}\right)}{\cos\frac{1}{x}\cdot\left(-\frac{1}{x^2}\right)} \qquad (LH)$$

$$= \lim_{x\to\infty} \frac{\sin\frac{1}{x}}{\cos\frac{1}{x}} = 0$$

$$\lim_{x\to\infty}(\cos\tfrac{1}{x})^x = e^0 = 1.$$

2. (5 points) Evaluate

$$\int_e^\infty \frac{1}{x(\ln x)^3}\,dx.$$

$$= \lim_{b\to\infty}\int_e^b \frac{1}{x(\ln x)^3}\,dx$$

$$= \lim_{b\to\infty}\int_e^b \frac{1}{(\ln x)^3}\,d(\ln x)$$

$$= \lim_{b\to\infty} -\frac{1}{2(\ln x)^2}\Big|_e^b$$

$$= \lim_{b\to\infty}\left(\frac{1}{2} - \frac{1}{2(\ln b)^2}\right) = 1/2.$$

3. (10 points) Determine whether the following series converges (state the name of your test and show how it is applied): (a) $\sum \frac{k!(2k)!}{(3k)!}$; (b) $\sum \frac{k^2}{k^{2.9}+10}$.

(a) Apply ~~ratio~~ ratio test

$$\lim_{k\to\infty} \frac{\frac{(k+1)!\,(2(k+1))!}{(3(k+1))!}}{\frac{k!\,(2k)!}{(3k)!}} = \lim_{k\to\infty} \frac{(k+1)(2k+2)(2k+1)}{(3k+3)(3k+2)(3k+1)}$$

$$= \frac{4}{27} < 1$$

So series (a) converges by ratio test.

(b) Compare with $\sum 1/k^{0.9}$

$$\lim_{k\to\infty} \frac{k^2/(k^{2.9}+10)}{1/k^{0.9}} = \lim \frac{k^{2.9}}{k^{2.9}+10} = 1$$

Since $\sum 1/k^{0.9}$ diverges (p-series), $\sum k^2/(k^{2.9}+10)$ also diverges by ~~the~~ limit comparison test.

4. (10 points) Find the interval of convergence of $\sum (-1)^k (x-3)^k/3^k$.

Apply root ~~test~~.

$$\lim_{k\to\infty} \sqrt[k]{\left|\frac{(-1)^k (x-3)^k}{3^k}\right|} = \frac{|x-3|}{3}$$

Set $\frac{|x-3|}{3} < 1$, we have $0 < x < 6$

When $x = 0$, the series is $\sum 1$ which diverges (divergence test)

When $x = 6$, the series is $\sum (-1)^k$ which diverges

So interval of convergence is $(0, 6)$.

5. (10 points) Use the Taylor series of $\sin x$ (in powers of x) to approximate $\int_0^1 \frac{\sin x}{x} dx$ within 0.001.

Taylor series $\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$

n^{th} remainder $R_n(x) = \frac{\sin^{(n+1)} c \; x^{n+1}}{(n+1)!}, \quad c \in (0, x)$

Note $\int_0^1 \frac{R_n(x)}{x} dx = \int_0^1 \frac{\sin^{(n+1)} c \cdot x^n}{(n+1)!} dx \leq \frac{1}{(n+1)!}$

The smallest n for $\frac{1}{(n+1)!} < 0.001$ is $n = 6$.

So $\int_0^1 \frac{\sin x}{x} dx \approx \int_0^1 \sum_{k=0}^{3} \frac{(-1)^k x^{2k}}{(2k+1)!} dx$

$= \int_0^1 \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!}\right) dx$

$= 1 - \frac{1}{3 \cdot 3!} + \frac{1}{5 \cdot 5!} - \frac{1}{7 \cdot 7!}$

6. (5 points) Solve $y' - y = e^{3x}$ with $y(0) = 1$.

$p(x) = -1, \; g(x) = e^{3x}$

$H(x) = \int p(x) dx = -x$

So $y = e^x \left[\int e^{-x} e^{3x} dx + C\right]$

$= e^x \left(\frac{1}{2} e^{2x} + C\right) = \frac{1}{2} e^{3x} + C e^x$

Since $y(0) = 1$, $C = \frac{1}{2}$

So $y = \frac{1}{2} e^{3x} + \frac{1}{2} e^x$.