Balanced judicious bipartitions of graphs

Baogang Xu [a,*], Juan Yan[a,b]
[a] School of Mathematics and Computer Science
Nanjing Normal University, 122 Ninghai Road, Nanjing, 210097, China
[b] College of Mathematics and System Sciences
Xinjiang University, Urumqi, Xinjiang 830046, China

Xingxing Yu[†]
School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332-0160, USA

Abstract

A bipartition of the vertex set of a graph is called *balanced* if the sizes of the sets in the bipartition differ by at most one. Bollobás and Scott [3] conjectured that if G is a graph with minimum degree at least 2 then $V(G)$ admits a balanced bipartition V_1, V_2 such that for each i, G has at most $|E(G)|/3$ edges with both ends in V_i. The minimum degree condition is necessary, and a result of Bollobás and Scott [5] shows that this conjecture holds for regular graphs G (i.e., when $\Delta(G) = \delta(G)$). We prove this conjecture for graphs G with $\Delta(G) \leq \frac{7}{5}\delta(G)$; hence it holds for graphs G with $\delta(G) \geq \frac{5}{7}|V(G)|$.

Key words and phrases: Balanced partition, judicious partition, maximum degree, minimum degree
AMS 2000 Subject Classifications: 05C35, 05C75

[*]Partially supported by NSFC 10671095. Email: baogxu@njnu.edu.cn
[†]Partially supported by NSA and by NSFC Project 16028102. Email: yu@math.gatech.edu

1 Introduction

The *Maximum Bipartite Subgraph Problem* is a classical partition problem which optimizes one quantity: Given a graph G, find a partition of $V(G)$ into V_1, V_2 that minimizes $e(V_1) + e(V_2)$, where $e(V_i)$ $(i \in \{1,2\})$ denotes the number of edges of G with both ends in V_i. A simple calculation shows that every graph with m edges has a bipartite subgraph with at least $m/2$ edges. Edwards [6,7] improved this lower bound to $\frac{m}{2} + \frac{1}{4}\sqrt{2m + \frac{1}{4}} - \frac{1}{8}$, which is essentially best possible as evidenced by the complete graphs K_{2n+1}. In [4] (also see [3]), Bollobás and Scott extend Edwards' bound to k-partitions of graphs by proving that the vertex set of any graph with m edges can be partitioned into $V_1, \ldots, V_k$ such that $e(V_1, \ldots, V_k) \geq \frac{k-1}{k}m + \frac{k-1}{2k}\sqrt{2m + \frac{1}{4}} + O(k^2)$, where $e(V_1, \ldots, V_k)$ denotes the number of edges of G that join vertices from different sets.

Judicious partition problems ask for a partition of the vertex set of a graph into subsets so that several quantities are optimized simultaneously. The *Bottleneck Bipartition Problem*, introduced by Entringer (see [10]), is such an example: Given a graph G, find a partition V_1, V_2 of $V(G)$ that minimizes $\max\{e(V_1), e(V_2)\}$. Székely and Shahrokhi [10] showed that this problem is NP-hard. Porter [8] proved that for any graph G with m edges there is a partition V_1, V_2 of $V(G)$ such that $\max\{e(V_1), e(V_2)\} \leq m/4 + O(\sqrt{m})$, establishing a conjecture of Erdös. (A matrix version of this Erdös conjecture was formulated by Entringer, and was solved by Porter and Székely [9].)

The Bottleneck Bipartition Problem was also studied by Bollobás and Scott [1,2]; they show in [2] that for any graph G with m edges there is a bipartition V_1, V_2 of $V(G)$ such that $e(V_1, V_2) \geq \frac{m}{2} + \frac{1}{4}\sqrt{2m + \frac{1}{4}} - \frac{1}{8}$ and $\max\{e(V_1), e(V_2)\} \leq \frac{m}{4} + \frac{1}{8}\sqrt{2m + \frac{1}{4}} - \frac{1}{16}$. Xu and Yu [11] extended this result to k-partitions (for $k \geq 3$), answering a question of Bollobás and Scott [3]: The vertex set of any graph with m edges can be partitioned into $V_1, \ldots, V_k$ such that $e(V_i) \leq \frac{m}{k^2} + \frac{k-1}{2k^2}\left(\sqrt{2m + \frac{1}{4}} - \frac{1}{2}\right)$ for $i \in \{1, 2, \ldots, k\}$, and $e(V_1, \ldots, V_k) \geq \frac{k-1}{k}m + \frac{1}{2k}\left(\sqrt{2m + \frac{1}{4}} - \frac{1}{2}\right)$.

This paper concerns the Bottleneck Bipartiton Problem with an additional requirement on the bipartitions. A k-partition $V_1, \ldots, V_k$ of $V(G)$ is said to be *balanced* if $-1 \leq |V_i| - |V_j| \leq 1$ for $1 \leq i, j \leq k$; the classical *Min k-Section Problem* asks for such a partition that minimizes $e(V_1, \ldots, V_k)$. Bollobás and Scott [3] asked an analogous question for judicious partitions: Given a graph G, find a balanced partition of $V(G)$ into $V_1, \ldots, V_k$ that minimizes $\max\{e(V_1), \ldots, e(V_k)\}$. In particular, they made the following conjecture, where $e(G)$ denotes the number of edges in the graph G.

Conjecture 1.1 (*Bollobás and Scott [3]*) *Let G be a graph with minimum degree at least 2. Then $V(G)$ admits a balanced partition V_1, V_2 such that $e(V_i) \leq e(G)/3$ for $i \in \{1, 2\}$.*

The complete graph K_3 shows that the bound $e(G)/3$ is sharp. The star $K_{1,n}$ shows that the requirement on minimum degree is necessary (otherwise, one cannot do better than $e(G)/2$ in general). Bollobás and Scott [5] proved the following result, which implies Conjecture 1.1 for regular graphs.

Theorem 1.2 (*Bollobás and Scott [5]*) *Let $d \geq 2$ be an integer, and let G be a d-regular graph. Then $V(G)$ admits a balanced bipartition V_1, V_2 such that*

(1) $e(V_i) \leq \frac{1}{4}\frac{d-1}{d}e(G)$ *when d is odd,*

(2) $e(V_i) \le \frac{1}{4}\frac{d}{d+1}e(G)$ *when d is even and $|V(G)|$ is even, and*

(3) $e(V_i) \le \frac{1}{4}\frac{d}{d+1}e(G) + \frac{d}{4}$ *when d is even and $|V(G)|$ is odd.*

Moreover, the extremal graphs for (1) *are sK_{d+1} for $s \ge 1$, those for* (2) *are $2sK_{d+1}$ for $s \ge 1$, and those for* (3) *are $(2s+1)K_{d+1}$ for $s \ge 0$.*

For a graph G, we use $\Delta(G)$ and $\delta(G)$ to denote the maximum and minimum degree of G, respectively. So a graph G is regular iff $\Delta(G) - \delta(G) = 0$. The following result of Yan and Xu [12] generalizes Theorem 1.2 to graphs G with $\Delta(G) - \delta(G) = 1$.

Theorem 1.3 (*Yan and Xu [12]*) *Let $d \ge 2$ be an integer, and let G be a graph with n_1 vertices of degree d and $n_2 := |V(G)| - n_1$ vertices of degree $d-1$. Then $V(G)$ admits a balanced bipartition V_1, V_2 such that*

(1) $e(V_i) \le e(G)/4 - n_1/8$ *when d is odd and $|V(G)|$ is even,*

(2) $e(V_i) \le e(G)/4 - n_1/8 + (d-1)/8$ *when d is odd and $|V(G)|$ is odd,*

(3) $e(V_i) \le e(G)/4 + n_2/8$ *when d is even and $|V(G)|$ is even,*

(4) $e(V_i) \le e(G)/4 + n_2/8 + d/8$ *when d is even and $|V(G)|$ is odd.*

The main goal of this paper is to provide further evidence to Conjecture 1.1, by proving it for graphs G for which $\Delta(G) - \delta(G)$ is not too large.

Theorem 1.4 *Let G be a graph, and assume that $\Delta(G) \le \frac{7}{5}\delta(G)$. Then G admits a balanced partition V_1, V_2 such that $e(V_i) \le e(G)/3$ for $i \in \{1,2\}$.*

Since $\Delta(G) \le |V(G)| - 1$, $\delta(G) \ge 5|V(G)|/7$ implies $\Delta(G) \le \frac{7}{5}\delta(G)$. So we have the following immediate consequence of Theorem 1.4, which implies Conjecture 1.1 for graphs G with $\delta(G) \ge 5|V(G)|/7$.

Corollary 1.5 *Let G be a graph with $\delta(G) \ge 5|V(G)|/7$. Then $V(G)$ admits a balanced partition V_1, V_2 such that $e(V_i) \le e(G)/3$ for $i \in \{1,2\}$.*

Theorems 1.2, 1.3 and 1.4 suggest that the bound on $e(V_i)$ in Conjecture 1.1 decrease from $e(G)/3$ to $e(G)/4$ as $\Delta(G)$ decreases from $\frac{7}{5}\delta(G)$ to $\delta(G)$. Indeed, the next result shows that this may be the case: The bound on $\max\{e(V_1), e(V_2)\}$ decreases from $e(G)/2$ to $e(G)/4$ as $\Delta(G)$ decreases from $3\delta(G)$ to $\delta(G)$. Note that $(r+4)/(3r-4)$ takes on the values $3, 7/5, 1$ when $r = 2, 3, 4$, respectively.

Theorem 1.6 *Let $2 \le r \le 4$ be a real number, and let G be a graph. Suppose $\Delta(G) \le \frac{r+4}{3r-4}\delta(G)$ when $|V(G)|$ is even, and $\Delta(G) \le \frac{r+4}{3r-4}\delta(G) - \frac{4r}{3r-4}$ when $|V(G)|$ is odd. Then $V(G)$ admits a balanced partition V_1, V_2 such that $e(V_i) \le e(G)/r$ for $i \in \{1,2\}$.*

The rest of this paper is organized as follows. In Section 2, we prove several lemmas. In Section 3 we prove Theorems 1.4 and 1.6. Section 4 contains remarks and further questions.

2 Lemmas

In this section, we prove three lemmas to be used in the proofs of Theorems 1.4 and 1.6. Let G be a graph and let V_1, V_2 be a partition of $V(G)$. For $j \in \{1,2\}$ and $i \in \{\delta(G), \delta(G)+1, \ldots, \Delta(G)\}$, we let $n_{j,i}$ denote the number of vertices in V_j that have degree i in G. When there is no possibility of confusion, we write δ and Δ instead of $\delta(G)$ and $\Delta(G)$.

Note that for $\delta \le i \le \Delta$, $0 \le \Delta - i \le \Delta - \delta$. We have the following simple observations for $j \in \{1,2\}$:

Observation (a). $\sum_{i=\delta}^{\Delta} n_{j,i} = \sum_{i=0}^{\Delta-\delta} n_{j,\Delta-i} = |V_j|$;

Observation (b). $\sum_{i=0}^{\Delta-\delta} i n_{j,\Delta-i} = \sum_{i=0}^{\Delta-\delta} \Delta n_{j,\Delta-i} - \sum_{i=0}^{\Delta-\delta} (\Delta - i) n_{j,\Delta-i} \le (\Delta-\delta)|V_j|$.

The first two lemmas express and estimate $e(V_i)$ in terms of $n_{j,i}$.

Lemma 2.1 *Let G be a graph, and let V_1, V_2 be a bipartition of $V(G)$. Then,*

(*i*) $e(G) = \frac{1}{2}\left(\Delta|V(G)| - \sum_{i=1}^{\Delta-\delta} i n_{1,\Delta-i} - \sum_{i=1}^{\Delta-\delta} i n_{2,\Delta-i}\right)$.

(*ii*) $e(V_1) - e(V_2) = \frac{1}{2}\sum_{i=1}^{\Delta-\delta} i(n_{2,\Delta-i} - n_{1,\Delta-i}) - \frac{\Delta}{2}(|V_2| - |V_1|)$.

Proof. By the Handshaking Lemma,

$$\begin{aligned} 2e(G) &= \sum_{i=\delta}^{\Delta} i(n_{1,i} + n_{2,i}) \\ &= \sum_{i=\delta}^{\Delta} \Delta(n_{1,i} + n_{2,i}) - \sum_{i=\delta}^{\Delta-1} (\Delta - i)(n_{1,i} + n_{2,i}) \\ &= \Delta(|V_1| + |V_2|) - \sum_{i=1}^{\Delta-\delta} i(n_{1,\Delta-i} + n_{2,\Delta-i}) \quad \text{(by Observation (a))} \\ &= \Delta|V(G)| - \sum_{i=1}^{\Delta-\delta} i n_{1,\Delta-i} - \sum_{i=1}^{\Delta-\delta} i n_{2,\Delta-i}, \end{aligned}$$

which proves (*i*). Since

$$2e(V_1) + e(V_1, V_2) = \sum_{i=\delta}^{\Delta} i n_{1,i}$$

and

$$2e(V_2) + e(V_1, V_2) = \sum_{i=\delta}^{\Delta} i n_{2,i},$$

$$
\begin{aligned}
e(V_1) - e(V_2) &= \frac{1}{2}\sum_{i=\delta}^{\Delta} i(n_{1,i} - n_{2,i}) \\
&= \frac{1}{2}\sum_{i=0}^{\Delta-\delta} (\Delta - i)(n_{1,\Delta-i} - n_{2,\Delta-i}) \\
&= \frac{1}{2}\left(\sum_{i=0}^{\Delta-\delta} i(n_{2,\Delta-i} - n_{1,\Delta-i}) + \Delta\sum_{i=0}^{\Delta-\delta} n_{1,\Delta-i} - \Delta\sum_{i=0}^{\Delta-\delta} n_{2,\Delta-i}\right).
\end{aligned}
$$

Therefore, (ii) follows from Observation (a). ∎

Lemma 2.2 *Let G be a graph, and let V_1, V_2 be a balanced bipartition of $V(G)$ such that $e(V_1, V_2)$ is maximum among all balanced bipartitions of $V(G)$. For $v \in V(G)$ let $t_v := |N(v) \cap V_1| - |N(v) \cap V_2|$, and let $t := \max\{t_v : v \in V_1\}$. Then*

(*i*) $e(V_1) \le \frac{\Delta+t}{4}|V_1| - \frac{1}{4}\sum_{i=1}^{\Delta-\delta} in_{1,\Delta-i}$.

(*ii*) $e(V_2) \le \frac{\Delta-t}{4}|V_2| - \frac{1}{4}\sum_{i=1}^{\Delta-\delta} in_{2,\Delta-i}$.

Proof. First, we estimate $e(V_1)$. Note that for $v \in V_1$, $t_v = |N(v) \cap V_1| - |N(v) \cap V_2| \le t$ and $|N(v) \cap V_1| + |N(v) \cap V_2| = d(v)$. So $|N(v) \cap V_1| \le \frac{d(v)+t}{2}$, and hence

$$
\begin{aligned}
2e(V_1) &= \sum_{v \in V_1} |N(v) \cap V_1| \\
&\le \frac{\Delta + t}{2} n_{1,\Delta} + \frac{(\Delta - 1) + t}{2} n_{1,\Delta-1} + \ldots + \frac{\delta + t}{2} n_{1,\delta} \\
&= \frac{\Delta + t}{2}|V_1| - \frac{1}{2}\sum_{i=1}^{\Delta-\delta} in_{1,\Delta-i} \quad \text{(by Observation (a))},
\end{aligned}
$$

which implies (*i*).

Next we estimate $e(V_2)$. Let $v_1 \in V_1$ with $t_{v_1} = t$.

Suppose for the moment that there exists $v_2 \in V_2$ such that $t_{v_2} = |N(v_2) \cap V_1| - |N(v_2) \cap V_2| < t = t_{v_1}$. Define $V_1' := (V_1 \setminus \{v_1\}) \cup \{v_2\}$ and $V_2' := (V_2 \setminus \{v_2\}) \cup \{v_1\}$. Then V_1', V_2' is also a balanced bipartition of $V(G)$, and

$$
\begin{aligned}
e(V_1', V_2') &\ge e(V_1, V_2) + (|N(v_1) \cap V_1| - |N(v_1) \cap V_2|) - (|N(v_2) \cap V_1| - |N(v_2) \cap V_2|) \\
&= e(V_1, V_2) + t_{v_1} - t_{v_2} \\
&\ge e(V_1, V_2) + 1,
\end{aligned}
$$

which contradicts the maximality of $e(V_1, V_2)$.

Therefore, for all $w \in V_2$, $t_w = |N(w) \cap V_1| - |N(w) \cap V_2| \geq t$. Since $|N(w) \cap V_1| + |N(w) \cap V_2| = d(w)$, we have $|N(w) \cap V_2| \leq \frac{d(w)-t}{2}$. Therefore,

$$\begin{aligned} 2e(V_2) &\leq \frac{\Delta - t}{2} n_{2,\Delta} + \frac{\Delta - 1 - t}{2} n_{2,\Delta-1} + \cdots + \frac{\delta - t}{2} n_{2,\delta} \\ &= \frac{\Delta - t}{2}|V_2| - \frac{1}{2}\sum_{i=1}^{\Delta-\delta} i n_{2,\Delta-i} \quad \text{(by Observation (a))}, \end{aligned}$$

which implies (ii). ∎

The next lemma implies Theorems 1.4 and 1.6 for graphs of even order. The technique is similar to that used in [5], by considering a balanced partition V_1, V_2 that maximizes $e(V_1, V_2)$.

Lemma 2.3 *Let $2 \leq r \leq 4$ be a real number, and let G be a graph such that $|V(G)|$ is even and $\Delta(G) \leq \frac{r+4}{3r-4}\delta(G)$. Then $V(G)$ admits a balanced bipartition V_1, V_2 such that $e(V_i) \leq e(G)/r$ for $i \in \{1, 2\}$.*

Proof. Let V_1, V_2 be a balanced bipartition of $V(G)$ such that $e(V_1, V_2)$ is maximum among all balanced bipartitions of $V(G)$. Then $|V_1| = |V_2| = |V(G)|/2$. Without loss of generality, we may assume that $e(V_1) \geq e(V_2)$. If $e(V_1) \leq e(G)/r$ then the assertion of the lemma holds. So we may assume that $e(V_1) > e(G)/r$.

Let $t_v := |N(v) \cap V_1| - |N(v) \cap V_2|$ (for all $v \in V(G)$) and define $t := \max\{t_v : v \in V_1\}$. By Lemma 2.2$(i)$ and the fact $|V_1| = |V(G)|/2$,

$$e(V_1) \leq \left(\frac{\Delta + t}{4}\right)\frac{|V(G)|}{2} - \frac{1}{4}\sum_{i=1}^{\Delta-\delta} i n_{1,\Delta-i},$$

where $\Delta := \Delta(G)$ and $\delta := \delta(G)$. By Lemma 2.1(i) and the assumption $e(V_1) > e(G)/r$,

$$\frac{1}{2r}\left(\Delta|V(G)| - \sum_{i=1}^{\Delta-\delta} i n_{1,\Delta-i} - \sum_{i=1}^{\Delta-\delta} i n_{2,\Delta-i}\right) < \left(\frac{\Delta + t}{4}\right)\frac{|V(G)|}{2} - \frac{1}{4}\sum_{i=1}^{\Delta-\delta} i n_{1,\Delta-i}.$$

Hence

$$\begin{aligned} &4\Delta|V(G)| \\ < \;& r(\Delta + t)|V(G)| - 2(r-2)\sum_{i=1}^{\Delta-\delta} i n_{1,\Delta-i} + 4\sum_{i=1}^{\Delta-\delta} i n_{2,\Delta-i} \\ \leq \;& r(\Delta + t)|V(G)| + 4\sum_{i=1}^{\Delta-\delta} i n_{2,\Delta-i} \quad \text{(since } r \geq 2\text{)} \\ \leq \;& r(\Delta + t)|V(G)| + 2(\Delta - \delta)|V(G)| \quad \text{(by Observation (b) and the fact } |V_2| = \tfrac{|V(G)|}{2}\text{)}. \end{aligned}$$

Therefore,

$$t > \frac{(2 - r)\Delta + 2\delta}{r}. \tag{1}$$

By Lemma 2.1(ii) and that fact $|V_1| = |V_2|$,

$$e(V_1) - e(V_2) = \frac{1}{2}\sum_{i=1}^{\Delta-\delta} i(n_{2,\Delta-i} - n_{1,\Delta-i}).$$

So it follows from Lemma 2.2(ii) and that fact $|V_2| = |V(G)|/2$ that

$$e(V_1) \le \left(\frac{\Delta - t}{4}\right)\frac{|V(G)|}{2} - \frac{1}{4}\sum_{i=1}^{\Delta-\delta} in_{2,\Delta-i} + \frac{1}{2}\sum_{i=1}^{\Delta-\delta} i(n_{2,\Delta-i} - n_{1,\Delta-i}).$$

Then, by Lemma 2.1(i) and the assumption $e(V_1) > e(G)/r$, we have

$$\begin{aligned} & \frac{1}{2r}\left(\Delta|V(G)| - \sum_{i=1}^{\Delta-\delta} in_{1,\Delta-i} - \sum_{i=1}^{\Delta-\delta} in_{2,\Delta-i}\right) \\ < \quad & \frac{\Delta - t}{4}\frac{|V(G)|}{2} - \frac{1}{4}\sum_{i=1}^{\Delta-\delta} in_{2,\Delta-i} + \frac{1}{2}\sum_{i=1}^{\Delta-\delta} i(n_{2,\Delta-i} - n_{1,\Delta-i}). \end{aligned}$$

Thus

$$\begin{aligned} & 4\Delta|V(G)| \\ < \quad & r(\Delta - t)|V(G)| + 2(r+2)\sum_{i=1}^{\Delta-\delta} in_{2,\Delta-i} - 4(r-1)\sum_{i=1}^{\Delta-\delta} in_{1,\Delta-i} \\ \le \quad & r(\Delta - t)|V(G)| + 2(r+2)\sum_{i=1}^{\Delta-\delta} in_{2,\Delta-i} \quad (\text{since } r \ge 2) \\ \le \quad & r(\Delta - t)|V(G)| + (r+2)(\Delta - \delta)|V(G)| \quad (\text{by Observation (b) and since } |V_2| = \tfrac{|V(G)|}{2}) \\ < \quad & \left(r\left(\Delta - \frac{(2-r)\Delta + 2\delta}{r}\right) + (r+2)(\Delta - \delta)\right)|V(G)| \quad (\text{by (1)}) \\ = \quad & (3r\Delta - (r+4)\delta)\,|V(G)|. \end{aligned}$$

Therefore,

$$\Delta > \frac{r+4}{3r-4}\delta,$$

a contradiction to the assumption that $\Delta \le \frac{r+4}{3r-4}\delta$. ∎

3 Proof of Theorems 1.4 and 1.6

Proof of Theorem 1.4. By Lemma 2.3 (with $r = 3$), we see that the assertion of Theorem 1.4 holds when $|V(G)|$ is even. So we may assume that $|V(G)|$ is odd.

Let V_1, V_2 be a balanced bipartition of $V(G)$ such that $e(V_1, V_2)$ is maximum among all balanced bipartitions of $V(G)$. Without loss of generality, we may assume that $e(V_1) \ge e(V_2)$. If $e(V_1) \le e(G)/3$, the assertion of Theorem 1.4 holds. So we may assume $e(V_1) > e(G)/3$. This, in particular, implies that $e(V_1, V_2) < 2e(G)/3$.

We claim that there exists $v_1 \in V_1$ such that $|N(v_1) \cap V_1| > |N(v_1) \cap V_2|$. For, otherwise, $|N(v) \cap V_1| \leq |N(v) \cap V_2|$ for all $v \in V_1$. Hence

$$\begin{aligned} 2e(V_1) &= \sum_{v \in V_1} |N(v) \cap V_1| \\ &\leq \sum_{v \in V_1} |N(v) \cap V_2| \\ &= e(V_1, V_2) \\ &< 2e(G)/3. \end{aligned}$$

This is a contradiction to the assumption that $e(V_1) > e(G)/3$.

Since V_1, V_2 is a balanced bipartition of $V(G)$, and since $n := |V(G)|$ is odd, either $|V_1| = \frac{n-1}{2}$ or $|V_1| = \frac{n+1}{2}$. Indeed,

$$|V_1| = \frac{n-1}{2} \quad \text{and} \quad |V_2| = \frac{n+1}{2}. \tag{2}$$

For, otherwise, $V_1' := V_1 \setminus \{v_1\}, V_2' := V_2 \cup \{v_1\}$ is also a balanced bipartition of $V(G)$, and $e(V_1', V_2') = e(V_1, V_2) + |N(v_1) \cap V_1| - |N(v_1) \cap V_2| \geq e(V_1, V_2) + 1$. But this contradicts the maximality of $e(V_1, V_2)$.

Let $t_v := |N(v) \cap V_1| - |N(v) \cap V_2|$ (for all $v \in V(G)$) and define $t := \max\{t_v : v \in V_1\}$. By Lemma 2.1($i$) and Lemma 2.2($i$), and by the assumption that $e(V_1) > e(G)/3$, we have

$$\frac{1}{3}(\Delta n - \sum_{i=1}^{\Delta-\delta} i n_{1,\Delta-i} - \sum_{i=1}^{\Delta-\delta} i n_{2,\Delta-i}) < \left(\frac{\Delta + t}{2}\right)\left(\frac{n-1}{2}\right) - \frac{1}{2}\sum_{i=1}^{\Delta-\delta} i n_{1,\Delta-i}.$$

Hence

$$\begin{aligned} \Delta n &< \frac{3}{4}(\Delta + t)(n-1) + \sum_{i=1}^{\Delta-\delta} i n_{2,\Delta-i} - \frac{1}{2}\sum_{i=1}^{\Delta-\delta} i n_{1,\Delta-i} \\ &\leq \frac{3}{4}(\Delta + t)(n-1) + \sum_{i=1}^{\Delta-\delta} i n_{2,\Delta-i} \\ &\leq \frac{3}{4}(\Delta + t)(n-1) + (\Delta - \delta)\frac{n+1}{2} \quad \text{(by Observation (b) and (2))} \\ &= \frac{3(n-1)t}{4} + \frac{(5n-1)\Delta - 2(n+1)\delta}{4}. \end{aligned}$$

Therefore

$$t > \frac{2(n+1)\delta - (n-1)\Delta}{3(n-1)} > \frac{2(n+1)\delta - (n-1)\Delta}{3n}. \tag{3}$$

By (2) and Lemma 2.1(ii),

$$e(V_1) - e(V_2) = \frac{1}{2}\sum_{i=1}^{\Delta-\delta} i(n_{2,\Delta-i} - n_{1,\Delta-i}) - \frac{\Delta}{2}.$$

So by Lemma 2.2(ii),

$$\begin{aligned} e(V_1) &\le \left(\frac{\Delta - t}{4}\right)\left(\frac{n+1}{2}\right) - \frac{1}{4}\sum_{i=1}^{\Delta-\delta} i n_{2,\Delta-i} + \frac{1}{2}\sum_{i=1}^{\Delta-\delta} i(n_{2,\Delta-i} - n_{1,\Delta-i}) - \frac{\Delta}{2} \\ &\le \left(\frac{\Delta - t}{4}\right)\frac{n}{2} - \frac{1}{4}\sum_{i=1}^{\Delta-\delta} i n_{2,\Delta-i} + \frac{1}{2}\sum_{i=1}^{\Delta-\delta} i(n_{2,\Delta-i} - n_{1,\Delta-i}) - \frac{3\Delta + t}{8}. \end{aligned}$$

Therefore, it follows from Lemma 2.1(i) and the assumption $e(V_1) > e(G)/3$ that

$$\frac{1}{3}(\Delta n - \sum_{i=1}^{\Delta-\delta} i n_{1,\Delta-i} - \sum_{i=1}^{\Delta-\delta} i n_{2,\Delta-i}) < \frac{(\Delta - t)n}{4} + \frac{1}{2}\sum_{i=1}^{\Delta-\delta} i n_{2,\Delta-i} - \sum_{i=1}^{\Delta-\delta} i n_{1,\Delta-i} - \frac{3\Delta + t}{4}.$$

By rearranging and combining terms, we have

$$\begin{aligned} \Delta n &< \frac{3n(\Delta - t)}{4} + \frac{5}{2}\sum_{i=1}^{\Delta-\delta} i n_{2,\Delta-i} - 2\sum_{i=1}^{\Delta-\delta} i n_{1,\Delta-i} - \frac{9\Delta + 3t}{4} \\ &\le \frac{3n(\Delta - t)}{4} + \frac{5}{2}\sum_{i=1}^{\Delta-\delta} i n_{2,\Delta-i} - \frac{9\Delta + 3t}{4} \\ &\le \frac{(3n-9)\Delta}{4} + \frac{5(n+1)(\Delta - \delta)}{4} - \frac{3(n+1)t}{4} \quad \text{(by (2) and Observation (b))} \\ &< \frac{(3n-9)\Delta}{4} + \frac{5(n+1)(\Delta - \delta)}{4} - \frac{3(n+1)}{4}\left(\frac{2(n+1)\delta - (n-1)\Delta}{3n}\right) \quad \text{(by (3))} \\ &= \frac{(3n-9)\Delta}{4} + \frac{5(n+1)(\Delta - \delta)}{4} - \frac{2(n+1)\delta - (n-1)\Delta}{4} - \frac{2(n+1)\delta - (n-1)\Delta}{4n} \\ &= \frac{4n\Delta - 10\Delta}{4} + \frac{5\Delta n + 5\Delta - 7\delta(n+1)}{4} - \frac{2(n+1)\delta - (n-1)\Delta}{4n} \\ &= \frac{9n\Delta - 7\delta(n+1)}{4} - \frac{2(n+1)\delta + 4n\Delta + \Delta}{4n} \\ &< \frac{9n\Delta - 7\delta(n+1)}{4}. \end{aligned}$$

Thus, $5n\Delta > 7(n+1)\delta > 7n\delta$. This implies $\Delta > 7\delta/5$, a contradiction. ∎

Proof of Theorem 1.6. By Lemma 2.3, the assertion of Theorem 1.6 holds when $|V(G)|$ is even. So we may assume that $n := |V(G)|$ is odd.

Let V_1, V_2 be a balanced bipartition of $V(G)$ such that $e(V_1, V_2)$ is maximum among all balanced bipartitions of $V(G)$. Assume, without loss of generality, that $e(V_1) \ge e(V_2)$. If $e(V_1) \le e(G)/r$ then the assertion of Theorem 1.6 holds. So we may assume that $e(V_1) > e(G)/r$.

Let $t_v := |N(v) \cap V_1| - |N(v) \cap V_2|$ (for all $v \in V(G)$), and define $t := \max\{t_v : v \in V_1\}$. Since $|V_1| = \frac{n-1}{2}$ or $|V_1| = \frac{n+1}{2}$, we consider two cases.

Case 1. $|V_1| = \frac{n+1}{2}$ and $|V_2| = \frac{n-1}{2}$.

We claim that $t \le 0$. For, if $t > 0$, then there is $v \in V_1$ such that $t_v > 0$. Now $V_1' := V_1 \setminus \{v\}, V_2' := V_2 \cup \{v\}$ is also a balanced bipartition of $V(G)$, and a simple calculation shows that $e(V_1', V_2') > e(V_1, V_2)$, contradicting the maximality of $e(V_1, V_2)$.

By Lemma 2.2(i), we have

$$e(V_1) \le \left(\frac{\Delta+t}{4}\right)\left(\frac{n+1}{2}\right) - \frac{1}{4}\sum_{i=1}^{\Delta-\delta} i n_{1,\Delta-i}.$$

Thus, by Lemma 2.1(i) and the assumption $e(V_1) > e(G)/r$,

$$\frac{1}{2r}\left(\Delta n - \sum_{i=1}^{\Delta-\delta} i n_{1,\Delta-i} - \sum_{i=1}^{\Delta-\delta} i n_{2,\Delta-i}\right) < \left(\frac{\Delta+t}{4}\right)\left(\frac{n+1}{2}\right) - \frac{1}{4}\sum_{i=1}^{\Delta-\delta} i n_{1,\Delta-i}.$$

Hence

$$\begin{aligned} 4\Delta n &< r(\Delta+t)(n+1) - 2(r-2)\sum_{i=1}^{\Delta-\delta} i n_{1,\Delta-i} + 4\sum_{i=1}^{\Delta-\delta} i n_{2,\Delta-i} \\ &\le r(\Delta+t)(n+1) + 4\sum_{i=1}^{\Delta-\delta} i n_{2,\Delta-i} \quad \text{(since } r \ge 2\text{)} \\ &\le r(\Delta+t)(n+1) + 2(\Delta-\delta)(n-1) \quad \text{(by Observation (b))}. \end{aligned} \tag{4}$$

By Lemma 2.2(ii), we have

$$e(V_2) \le \left(\frac{\Delta-t}{4}\right)\left(\frac{n-1}{2}\right) - \frac{1}{4}\sum_{i=1}^{\Delta-\delta} i n_{2,\Delta-i}.$$

By Lemma 2.1(ii),

$$e(V_1) - e(V_2) = \frac{1}{2}\sum_{i=1}^{\Delta-\delta} i(n_{2,\Delta-i} - n_{1,\Delta-i}) + \frac{\Delta}{2}.$$

These two expressions imply

$$e(V_1) \le \left(\frac{\Delta-t}{4}\right)\left(\frac{n-1}{2}\right) + \frac{1}{4}\sum_{i=1}^{\Delta-\delta} i n_{2,\Delta-i} - \frac{1}{2}\sum_{i=1}^{\Delta-\delta} i n_{1,\Delta-i} + \frac{\Delta}{2}.$$

Thus, by Lemma 2.1(i) and the assumption $e(V_1) > e(G)/r$,

$$\frac{1}{2r}\left(\Delta n - \sum_{i=1}^{\Delta-\delta} i n_{1,\Delta-i} - \sum_{i=1}^{\Delta-\delta} i n_{2,\Delta-i}\right) < \left(\frac{\Delta-t}{4}\right)\left(\frac{n-1}{2}\right) + \frac{1}{4}\sum_{i=1}^{\Delta-\delta} i n_{2,\Delta-i} - \frac{1}{2}\sum_{i=1}^{\Delta-\delta} i n_{1,\Delta-i} + \frac{\Delta}{2}.$$

Hence,

$$\begin{aligned} 4\Delta n &< r(\Delta-t)(n-1) + (2r+4)\sum_{i=1}^{\Delta-\delta} i n_{2,\Delta-i} - (4r-4)\sum_{i=1}^{\Delta-\delta} i n_{1,\Delta-i} + 4r\Delta \\ &\le r(\Delta-t)(n-1) + (2r+4)\sum_{i=1}^{\Delta-\delta} i n_{2,\Delta-i} + 4r\Delta \quad \text{(since } r \ge 2\text{)} \\ &\le r(\Delta-t)(n-1) + (r+2)(\Delta-\delta)(n-1) + 4r\Delta \quad \text{(by Observation (b))}. \end{aligned} \tag{5}$$

Since $\Delta \le n-1$, $4(n+1)r\Delta \le 4(n^2-1)r$. Multiplying (4) by $n-1$ and (5) by $n+1$, and combining the resulting inequalities, we have

$$\begin{aligned}
8\Delta n^2 &< 2r\Delta(n^2-1) + 2(\Delta-\delta)(n-1)^2 + (r+2)(\Delta-\delta)(n^2-1) + 4(n+1)r\Delta \\
&\le 3r\Delta n^2 + 4n^2\Delta - (r+4)n^2\delta - 4n(\Delta-\delta) - (3r\Delta - r\delta) + 4(n^2-1)r \\
&= 3rn^2\Delta + 4n^2\Delta - (r+4)n^2\delta + 4rn^2 - 4n(\Delta-\delta) - (3r\Delta - r\delta + 4r) \\
&\le 3rn^2\Delta + 4n^2\Delta - (r+4)n^2\delta + 4rn^2.
\end{aligned}$$

Therefore, $\Delta > \frac{r+4}{3r-4}\delta - \frac{4r}{3r-4}$, a contradiction.

Case 2. $|V_1| = \frac{n-1}{2}$ and $|V_2| = \frac{n+1}{2}$.

By Lemma 2.1(ii),

$$e(V_1) - e(V_2) = \frac{1}{2}\sum_{i=1}^{\Delta-\delta} i(n_{2,\Delta-i} - n_{1,\Delta-i}) - \frac{\Delta}{2}.$$

By Lemma 2.2(ii),

$$e(V_2) \le \left(\frac{\Delta-t}{4}\right)\left(\frac{n+1}{2}\right) - \frac{1}{4}\sum_{i=1}^{\Delta-\delta} in_{2,\Delta-i}.$$

These two expressions imply

$$e(V_1) \le \left(\frac{\Delta-t}{4}\right)\left(\frac{n+1}{2}\right) + \frac{1}{4}\sum_{i=1}^{\Delta-\delta} in_{2,\Delta-i} - \frac{1}{2}\sum_{i=1}^{\Delta-\delta} in_{1,\Delta-i} - \frac{\Delta}{2}.$$

Hence, by Lemma 2.1(i) and the assumption $e(V_1) > e(G)/r$, we have

$$\frac{1}{2r}\left(\Delta n - \sum_{i=1}^{\Delta-\delta} in_{1,\Delta-i} - \sum_{i=1}^{\Delta-\delta} in_{2,\Delta-i}\right) < \left(\frac{\Delta-t}{4}\right)\left(\frac{n+1}{2}\right) + \frac{1}{4}\sum_{i=1}^{\Delta-\delta} in_{2,\Delta-i} - \frac{1}{2}\sum_{i=1}^{\Delta-\delta} in_{1,\Delta-i} - \frac{\Delta}{2}.$$

So

$$\begin{aligned}
4n\Delta &< r(\Delta-t)(n+1) + 2(r+2)\sum_{i=1}^{\Delta-\delta} in_{2,\Delta-i} - (4r-4)\sum_{i=1}^{\Delta-\delta} in_{1,\Delta-i} - 4r\Delta \\
&\le r(\Delta-t)(n+1) + 2(r+2)\sum_{i=1}^{\Delta-\delta} in_{2,\Delta-i} - 4r\Delta \quad \text{(since } r \ge 2\text{)} \\
&\le r(\Delta-t)(n+1) + (r+2)(\Delta-\delta)(n+1) - 4r\Delta \quad \text{(by Observation (b)).} \qquad (6)
\end{aligned}$$

By Lemma 2.2(i),

$$e(V_1) \le \left(\frac{\Delta+t}{4}\right)\left(\frac{n-1}{2}\right) - \frac{1}{4}\sum_{i=1}^{\Delta-\delta} in_{1,\Delta-i}.$$

Therefore, by the assumption $e(V_1) > e(G)/r$ and by Lemma 2.1(i), we have

$$\frac{1}{2r}\left(\Delta n - \sum_{i=1}^{\Delta-\delta} i n_{1,\Delta-i} - \sum_{i=1}^{\Delta-\delta} i n_{2,\Delta-i}\right) < \left(\frac{\Delta+t}{4}\right)\left(\frac{n-1}{2}\right) - \frac{1}{4}\sum_{i=1}^{\Delta-\delta} i n_{1,\Delta-i}.$$

So

$$\begin{aligned} 4n\Delta &< r(\Delta+t)(n-1) - 2(r-2)\sum_{i=1}^{\Delta-\delta} i n_{1,\Delta-i} + 4\sum_{i=1}^{\Delta-\delta} i n_{2,\Delta-i} \\ &\leq r(\Delta+t)(n-1) + 4\sum_{i=1}^{\Delta-\delta} i n_{2,\Delta-i} \quad \text{(since } r \geq 2) \\ &\leq r(\Delta+t)(n-1) + 2(\Delta-\delta)(n+1) \quad \text{(by Observation (b))}. \end{aligned} \tag{7}$$

Multiplying (7) by $(n+1)$ and (6) by $(n-1)$, and combining the resulting inequalities, we get

$$\begin{aligned} 8n^2\Delta &< 2r(n^2-1)\Delta + 2(\Delta-\delta)(n+1)^2 + (r+2)(\Delta-\delta)(n^2-1) - 4(n-1)r\Delta \\ &= (3rn^2 + 4n^2 + 4n + r - 4nr)\Delta - \left((r+4)n^2 + 4n - r\right)\delta. \end{aligned}$$

Hence

$$\left((r+4)n^2 + 4n - r\right)\delta < (3r-4)n^2\Delta + (4n + r - 4rn)\Delta,$$

and so, $(r+4)n^2\delta < (3r-4)n^2\Delta$. This implies $\Delta > \frac{r+4}{3r-4}\delta$, a contradiction. ∎

4 Further discussions

The proofs of Lemma 2.3 and Theorems 1.4 and 1.6 actually show that for any graph G with $\Delta(G) \leq \frac{7}{5}\delta(G)$, any balanced bipartition V_1, V_2 of $V(G)$ with $e(V_1, V_2)$ maximum (among all balanced bipartitions) must satisfy $e(V_i) \leq e(G)/3$. (The maximality of the partition makes it possible to derive the bound on $e(V_i)$, by allowing us to exchange some vertex of V_1 with a vertex of V_2.) Unfortunately, this is not always the case. For the graph G in Figure 1, the bipartition $V_1 := \{x_1, \ldots, x_7\}, V_2 := \{y_1, \ldots, y_7\}$ of $V(G)$ is the unique balanced bipartition of $V(G)$ for which $e(V_1, V_2)$ is maximum. However, $e(V_1) = 15 > 44/3 = e(G)/3$. Since it is not obvious why the partition V_1, V_2 is the unique maximum balanced bipartition of $V(G)$, we give a proof of this fact; which in a way indicates that when dealing with balanced bipartitions for general graphs, it is necessary to exchange subsets (of V_i) of size more than one.

Note that G has a "reflection" symmetry in the line through the edge x_4y_4. Also note that $e(V_2) = 0$, $e(V_1) = 15$, and $e(V_1, V_2) = 29$.

Let V_1', V_2' be an arbitrary balanced bipartition of $V(G)$ different from V_1, V_2. Then there exist $S_i \subseteq V_i$, $i = 1, 2$, with $0 \neq |S_1| = |S_2| \leq 3$ such that $V_i' = (V_i \setminus S_i) \cup S_{3-i}$. We now proceed to show that $e(V_1', V_2') < e(V_1, V_2)$. Observe that

$$\begin{aligned} & e(V_1', V_2') \\ =\ & e((V_1 \setminus S_1) \cup S_2, (V_2 \setminus S_2) \cup S_1) \\ =\ & e(V_1, V_2) - e(S_1, V_2 \setminus S_2) - e(S_2, V_1 \setminus S_1) + e(S_1, V_1 \setminus S_1). \end{aligned}$$

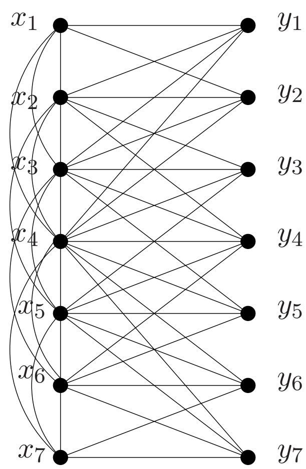

Figure 1: A graph with a unique maximum balanced bipartition.

So it suffices to show that

$$e(S_1, V_1 \setminus S_1) - e(S_1, V_2 \setminus S_2) < e(S_2, V_1 \setminus S_1).$$

Let $t_j := |N(x_j) \cap V_1| - |N(x_j) \cap V_2|$ for $1 \leq j \leq 7$, and let $t(S_1) := \sum_{x_j \in S_1} t_j$. Then

$$\begin{aligned}
& e(S_1, V_1 \setminus S_1) - e(S_1, V_2 \setminus S_2) \\
= & \left(\sum_{x_j \in S_1} |N(x_j) \cap V_1| \right) - 2e(S_1) - \left(\left(\sum_{x_j \in S_1} |N(x_j) \cap V_2| \right) - e(S_1, S_2) \right) \\
= & \left(\sum_{x_j \in S_1} t_j \right) - 2e(S_1) + e(S_1, S_2) \\
= & \ t(S_1) - 2e(S_1) + e(S_1, S_2).
\end{aligned}$$

Thus, it suffices to show that

$$t(S_1) - 2e(S_1) + e(S_1, S_2) < e(S_2, V_1 \setminus S_1).$$

We now list a few useful observations about the graph G:

(1) $t_j = 0$ for $j \in \{2, 3, 5, 6\}$, $t_1 = t_7 = 1$, and $t_4 = -1$;

(2) $-1 \leq t(S_1) \leq 2$;

(3) $t(S_1) = 2$ iff $\{x_1, x_7\} \subseteq S_1$ and $x_4 \notin S_1$;

(4) $t(S_1) = -1$ iff $x_4 \in S_1$ and $\{x_1, x_7\} \cap S_1 = \emptyset$;

(5) $e(S_2, V_1 \setminus S_1) \geq 4|S_2| - e(S_2, S_1)$.

If $|S_1| = |S_2| = 1$, then $e(S_1, S_2) \leq 1$, $e(S_1) = 0$, and $t(S_1) \leq 1$ (by (1)). Hence $t(S_1) - 2e(S_1) + e(S_1, S_2) \leq 1 - 0 + 1 < 4 - 1 \leq e(S_2, V_1 \setminus S_1)$ (by (5)). So we may assume $|S_1| = |S_2| \in \{2, 3\}$.

Case 1. $|S_1| = |S_2| = 2$.

Then $e(S_1, S_2) \leq 4$ and $e(S_1) \leq 1$.

Suppose $e(S_1) = 1$. Then $S_1 \not\supseteq \{x_1, x_7\}$ (since $x_1x_7 \notin E(G)$). It follows from (2) and (3) that $t(S_1) \leq 1$. Hence $t(S_1) - 2e(S_1) + e(S_1, S_2) \leq 1 - 2 + 4 < e(S_2, V_1 \setminus S_1)$ (by (5)).

Now assume $e(S_1) = 0$. Then $x_4 \notin S_1$ since x_4 is adjacent to all vertices (including those in $S_1 \setminus \{x_1\}$; thus $t(S_1) \neq -1$ (by (4)).

If $t(S_1) = 2$, then $S_1 = \{x_1, x_7\}$ (by (3)). Since x_1 and x_7 have no common neighbor in S_2, $e(S_1, S_2) \leq 2$. Therefore $t(S_1) - 2e(S_1) + e(S_1, S_2) \leq 2 - 0 + 2 < e(S_2, V_1 \setminus S_1)$ (by (5)).

Assume $t(S_1) = 1$. By (1), $x_4 \notin S_1$, $S_1 \cap \{x_1, x_7\} \neq \emptyset$ and $\{x_2, x_3, x_5, x_6\} \cap S_1 \neq \emptyset$. By symmetry, we may assume that $x_1 \in S_1$. Since $e(S_1) = 0$, $S_1 = \{x_1, x_5\}$ or $S_1 = \{x_1, x_6\}$. Hence $e(S_1, S_2) \leq 2$, and so $t(S_1) - 2e(S_1) + e(S_1, S_2) \leq 1 - 0 + 2 < e(S_2, V_1 \setminus S_1)$ (by (5)).

So we may assume $t(S_1) = 0$. Since $e(S_1) = 0$ and $x_4 \notin S_1$ and by (1), we have $S_1 = \{x_2, x_6\}$. So $e(S_1, S_2) \leq 3$ (since $|N(x_2) \cap N(x_6) \cap V_2| = 1$), and hence $t(S_1) - 2e(S_1) + e(S_1, S_2) \leq 0 - 0 + 3 < e(S_2, V_1 \setminus S_1)$ (by (5)).

Case 2. $|S_1| = |S_2| = 3$.

Then $e(S_1, S_2) \leq 9$ and $e(S_1) \leq 3$. Also note that $e(S_1) \geq 1$.

First assume $e(S_1) = 3$. Then, $\{x_1, x_7\} \not\subseteq S_1$, and hence $t(S_1) \leq 1$ by (2) and (3). If $t(S_1) = -1$, then $t(S_1) - 2e(S_1) + e(S_1, S_2) \leq -1 - 6 + 9 < e(S_2, V_1 \setminus S_1)$ (by (5)). So we assume $t(S_1) \geq 0$. It suffices to show $e(S_1, S_2) \leq 8$, since in that case $t(S_1) - 2e(S_1) + e(S_1, S_2) \leq 1 - 6 + 8 < e(S_2, V_1 \setminus S_1)$ (by (5)). This is clear if $\{x_1, x_7\} \cap S_1 \neq \emptyset$, since x_1 and x_7 each have just two neighbors in V_2. So we may assume $\{x_1, x_7\} \cap S_1 = \emptyset$. Then $t(S_1) = 0$, and $S_1 \subseteq \{x_2, x_3, x_5, x_6\}$. Since $e(S_1) = 3$ and $x_2x_6 \notin E(G)$, we may assume by symmetry that $S_1 = \{x_2, x_3, x_5\}$. Then $e(S_1, S_2) \leq 8$, since $|N(x_2) \cap N(x_5) \cap V_2| = 2$.

Now assume $e(S_1) = 2$. Then $t(S_1) \leq 1$; otherwise by (2) and (3), $\{x_1, x_7\} \subseteq S_1$ and $x_4 \notin S_1$, and we would have $e(S_1) \leq 1$. So it suffices to show that $e(S_1, S_2) \leq 7$, in which case $t(S_1) - 2e(S_1) + e(S_1, S_2) \leq 1 - 4 + 7 < e(S_2, V_1 \setminus S_1)$ (by (5)). If $t(S_1) = -1$ then by (4) and since $e(S_1) = 2$, we have $S_1 = \{x_2, x_4, x_6\}$, and so $e(S_1, S_2) \leq 7$ (since $|N(x_2) \cap N(x_6) \cap V_2| = 1$). Suppose $t(S_1) = 1$. Then by (1), $\{x_1, x_7\} \cap S_1 \neq \emptyset$. So by symmetry assume $x_1 \in S_1$. If $x_4 \in S_1$, again by (1), $S_1 = \{x_1, x_4, x_7\}$, and so $e(S_1, S_2) \leq 7$ (since $|N(x_1) \cap N(x_7) \cap V_2| = 0$). So assume $x_4 \notin S_1$. Then $x_7 \notin S_1$, and so, $S_1 = \{x_1, x_2, x_5\}$ or $S_1 = \{x_1, x_3, x_5\}$ or $S_1 = \{x_1, x_3, x_6\}$. In these cases we have $e(S_1, S_2) \leq 7$ (since $|N(x_1) \cap N(x_5) \cap V_2| = |N(x_1) \cap N(x_6) \cap V_2| = 0$). Now suppose $t(S_1) = 0$. If $x_4 \in S_1$, then by (1) and since $e(S_1) = 2$, exactly one of $\{x_1, x_7\}$, say x_1 (by symmetry), is in S_1; thus $S_1 = \{x_1, x_4, x_5\}$ or $S_1 = \{x_1, x_4, x_6\}$, and hence $e(S_1, S_2) \leq 7$ (since $|N(x_1) \cap N(x_5) \cap V_2| = |N(x_1) \cap N(x_6) \cap V_2| = 0$). So $x_4 \notin S_1$. Then since $t(S_1) = 0$ and by (1), $\{x_1, x_7\} \cap S_1 = \emptyset$. Hence, since $e(S_1) = 2$, $S_1 = \{x_2, x_3, x_6\}$ or $S_1 = \{x_6, x_5, x_2\}$, and we have $e(S_1, S_2) \leq 7$ again (since $|N(x_2) \cap N(x_6) \cap V_2| = 1$).

Finally assume $e(S_1) = 1$. Then $x_4 \notin S_1$, and so $t(S_1) \neq -1$ (by (4)). Moreover, $t(S_1) \neq 0$ as otherwise $S_1 \subseteq \{x_2, x_3, x_5, x_6\}$ which implies $e(S_1) \geq 2$. So $1 \leq t(S_1) \leq 2$. If $t(S_1) = 2$ then by (1), $S_1 = \{x_1, x_7, x_k\}$, with $k \in \{2, 3, 5, 6\}$; in these cases we can check that $e(S_1, S_2) \leq 5$, and so $t(S_1) - 2e(S_1) + e(S_1, S_2) \leq 2 - 2 + 5 < e(S_2, V_1 \setminus S_1)$ (by (5)). If $t(S_1) = 1$ then by (1), exactly one of $\{x_1, x_7\}$, say x_1 (by symmetry), belongs to S_1. Since $e(S_1) = 1$, $S_1 = \{x_1, x_2, x_6\}$, and so $e(S_1, S_2) \leq 6$. Hence $t(S_1) - 2e(S_1) + e(S_1, S_2) \leq 1 - 2 + 6 < e(S_2, V_1 \setminus S_1)$

(by (5)).

Therefore, we have shown that V_1, V_2 is the unique balanced bipartition of $V(G)$ such that $e(V_1, V_2)$ is maximal among all such partitions. So the constant c in the following question satisfies $7/5 \leq c < 13/4$.

Problem 4.1 *What is the largest constant c such that for any graph G with $\Delta(G) \leq c\delta(G)$, if V_1, V_2 is a balanced bipartition of $V(G)$ with $e(V_1, V_2)$ maximum then* $\max\{e(V_1), e(V_2)\} \leq e(G)/3$*?*

We conclude this paper with the following question of Bollobás and Scott.

Problem 4.2 (*Bollobás and Scott [3]*) *What is the smallest constant $c(d)$ such that every graph G with $\delta(G) \geq d$ has a balanced bipartition V_1, V_2 such that* $\max\{e(V_1), e(V_2)\} \leq c(d)e(G)$*?*

Acknowledgment. We thank an anonymous referee for pointing us to reference [9].

References

[1] B. Bollobás and A. D. Scott, Judicious partitions of graph, *Period. Math. Hungar* **26** (1993) 125–137.

[2] B. Bollobás and A. D. Scott, Exact bounds for judicious partitions of graphs, *Combinatorica* **19** (1999) 473–486.

[3] B. Bollobás and A. D. Scott, Problems and results on judicious partitions, *Random Struct. Alg.* **21** (2002) 414–430.

[4] B. Bollobás and A. D. Scott, Better bounds for Max Cut, in Contemporary Comb, Bolyai Soc Math Stud 10, *János Bolyai Math Soc*, Budapest, 2002, pp. 185–246.

[5] B. Bollobás and A. D. Scott, Judicious partitions of bounded-degree graphs, *J Graph Theory* **46** (2004) 131–143.

[6] C. S. Edwards, Some extremal properties of bipartite graphs, *Canadian J. math.* **25** (1973) 475–485.

[7] C. S. Edwards, An improved lower bound for the number of edges in a largest bipartite subgraph, in *Proc. 2nd Czechoslovak Symposium on Graph Theory*, Prague (1975) 167–181.

[8] T. D. Porter, On a bottleneck bipartition conjecture of Erdös, *Combinatorica* **12** (1992) 317–321.

[9] T. D. Porter and L. A. Székely, On a matrix discrepancy problem, *Congressus Numerantium* **73** (1990) 239–248.

[10] F. Shahrokhi and L. A. Székely, The complexity of the bottleneck graph bipartition problem, *J. Combin. Math. Combin. Comp.* **15** (1994) 221–226.

[11] B. Xu and X. Yu, Judicious k-partitions of graphs, manuscript submitted.

[12] J. Yan and B. Xu, Balanced judicious partitions of $(k, k-1)$-biregular graphs, manuscript submitted.