On Judicious Bisections of Graphs

Baogang Xu*
Institute of Mathematics
School of Mathematical Sciences
Nanjing Normal University
1 Wenyuan Road, Nanjing, 210023, China
Email: baogxu@njnu.edu.cn

Xingxing Yu†
School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332-0160, USA
Email:yu@math.gatech.edu

December 9, 2013

Abstract

A *bisection* of a graph G is a bipartition S_1, S_2 of $V(G)$ such that $-1 \leq |S_1| - |S_2| \leq 1$. It is NP-hard to find a bisection S_1, S_2 of a graph G maximizing $e(S_1, S_2)$ (respectively, minimizing $\max\{e(S_1), e(S_2)\}$), where $e(S_1, S_2)$ denotes the number of edges of G between S_1 and S_2, and $e(S_i)$ denotes the number of edges of G with both ends in S_i. There has been algorithmic work on bisections, but very few extremal results are known. Bollobás and Scott conjectured that if G is a graph with m edges and minimum degree at least 2 then G admits a bisection S_1, S_2 such that $\max\{e(S_1), e(S_2)\} \leq m/3$. In this paper, we confirm this conjecture and show that the triangle is the only extremal graph. Moreover, the bound $m/3$ cannot be improved to $(1/3 - \epsilon)m$, for any $\epsilon > 0$, by excluding K_3 or by increasing the minimum degree from 2 to 3.

Key words and phrases: Balanced partition, judicious partition
AMS 2000 Subject Classifications: 05C35, 05C75

*Partially supported by NSFC projects 10931003 and 11171160 and by the Doctoral Fund of Ministry of Education of China.
†Partially supported by NSF grants DMS-1265524 and AST-1247545 and NSA grant H98230-13-1-0255

1 Introduction

Let G be a graph and k a positive integer. A *k-partition* of G is a partition of $V(G)$ into k pairwise disjoint nonempty sets. Let $S_1, \ldots, S_k$ be a k-partition of G. For $1 \le i \le k$, we use $e(S_i)$ to denote the number of edges of G with both ends in S_i. Let $e(G)$ denote the number of edges of G, and let $e(S_1, \ldots, S_k) = e(G) - \sum_{i=1}^{k} e(S_i)$ denote the *size* of this partition. A 2-partition of G is also called a *bipartition* of G, and is often denoted as $[S, \overline{S}]$, where $S \subseteq V(G)$ and $\overline{S} = V(G) \setminus S$.

The *Maximum Bipartite Subgraph Problem* asks for a bipartition $[S, \overline{S}]$ of a graph that maximizes $e(S, \overline{S})$, which is the unweighted version of the famous Max-Cut problem. This problem is NP-hard (see [11]), even when restricted to graphs with maximum degree 3 (see [22]). There has been extensive body of results on this problem as well as related problems, both extremal and algorithmic (see [14]). One of the most notable result on this problem is due to Goemans and Williamson [10] who found an SDP based approximation algorithm with performance guarantee 0.87856, which is optimal assuming the Unique Games Conjecture (see [12]). On the extremal side, Edwards [7,8] proved that every graph with m edges admits a bipartition of size at least $m/2 + (1/4)(\sqrt{2m + 1/4} - 1/2)$, which is sharp for complete graphs of odd order. Bollobás and Scott [4] generalized this result by showing that every graph with m edges admits a k-partition of size at least

$$\frac{k-1}{k} m + \frac{k-1}{2k} \left(\sqrt{2m + 1/4} - 1/2 \right) + O(k). \tag{1.1}$$

In many situations, one may need to deal with *judicious partitioning problems* which seek for partitions that optimize several quantities simultaneously. An example is the *Bottleneck Bipartition Problem* that asks for a bipartition $[S, \overline{S}]$ of a graph G to minimize $\max\{e(S), e(\overline{S})\}$. This problem was shown to be NP-hard by Székely and Shahrokhi [17]. Let G be a graph with m edges. Porter [15] proved that there is a bipartition $[S, \overline{S}]$ of G such that $\max\{e(S), e(\overline{S})\} \le m/4 + O(\sqrt{m})$, confirming a conjecture of Erdős. Bollobás and Scott [2] improved and generalized Porter's result by showing that for each integer $k \ge 2$, G admits a k-partition $S_1, \ldots, S_k$ such that

$$\max_{1 \le i \le k} \{e(S_i)\} \le \frac{1}{k^2} m + \frac{k-1}{2k^2} \left(\sqrt{2m + 1/4} - 1/2 \right), \tag{1.2}$$

and that K_{kn+1} are the only extremal graphs (modulo isolated vertices). They also conjectured that G admits a k-partition $S_1, \ldots, S_k$ satisfying both (1.1) and (1.2), which was confirmed by the present authors [20,21]. We refer the reader to [3,16] for more problems and results on judicious partitioning problems.

Alon, Bollobás, Krivelevich and Sudakov [1] proved that there is a useful connection between the Maximum Bipartite Subgraph Problem and the Bottleneck Bipartition Problem: Graphs with large bipartite subgraphs also have good judicious bipartitions. Bollobás and Scott [5] extended this result to give a relation between maximum k-cuts and judicious k-partitions.

In this paper, we will study bipartitions that are balanced. A k-partition $S_1, \ldots, S_k$ of a graph is said to be *balanced* if $-1 \le |S_i| - |S_j| \le 1$ for all $i, j \in \{1, \ldots, k\}$; and a balanced bipartition is also known as a *bisection*. Graph bisection problems have been studied algorithmically. For example, Feige and Langberg [9] gave an approximation algorithm with performance guarantee 0.7028. Note that this guarantee is much smaller than that of Goemans and Williamson for Max-Cut, an indication that the Maximum Bisection Problem might be "harder" than the Max-Cut Problem. Enhancing this view, Lee, Loh and Sudakov [13] recently demonstrated that there is no connection similar to that in [1] between maximum bisections and judicious bisections.

As pointed out by Bollobás and Scott [3], the extremal problems for balanced partitions have been relatively little investigated. However there has been recent work on graph bisections, see [13, 18, 19], partly motivated by the following conjecture of Bollobás and Scott [3]: Every graph with m edges and minimum degree at least 2 admits a *good* bisection, i.e., a bisection $[S, \overline{S}]$ such that $\max\{e(S), e(\overline{S})\} \leq m/3$. The star $K_{1,n}$ shows that the requirement on minimum degree is necessary, and the triangle shows that the bound is best possible.

Bollobás and Scott [6] proved that for large m, regular graphs with m edges admit bisections $[S, \overline{S}]$ such that $\max\{e(S), e(\overline{S})\} < m/4$. Xu, Yan and Yu [18] proved that if $\Delta(G) \leq 7\delta(G)/5$ then every maximum bisection of G is a good bisection, where $\Delta(G)$ and $\delta(G)$ denote the maximum and minimum degrees of G, respectively. On the other hand, Lee, Loh and Sudakov [13] proved that there are graphs with large bisections but with no bisections close to being good. They also proved that every graph with m edges and minimum degree $2k$ or $2k+1$ admits a bisection $[S, \overline{S}]$ such that $\max\{e(S), e(\overline{S})\} \leq (\frac{k+1}{2(2k+1)} + o(1))m$, which asymptotically answers a question of Bollobás and Scott [3] concerning the dependence on $\delta(G)$ of judicious bounds on bisections. It remains an interesting question whether this result holds with the $o(1)$ term removed.

The main result of this paper is the following, which confirms the Bollobás–Scott conjecture mentioned above.

Theorem 1.1. *Every graph with m edges and minimum degree at least 2 admits a good bisection, i.e., a bisection $[S, \overline{S}]$ such that $\max\{e(S), e(\overline{S})\} \leq m/3$. Moreover, the triangle K_3 is the only extremal graph.*

The following examples show that the bound $m/3$ cannot be improved to $(1/3 - \epsilon)m$, for any $\epsilon > 0$, by excluding K_3 or by increasing the minimum degree from 2 to 3. In $K_{3,k}$, every bisection $[S, \overline{S}]$ has $\max\{e(S), e(\overline{S})\} \geq k - 2 = (1/3 - 2/3k)e(K_{3,k})$. Let F_k denote the graph obtained from k vertex disjoint triangles by identifying one vertex from each triangle to a single vertex. It is easy to check that if $[S, \overline{S}]$ is a bisection of F_k then $\max\{e(S), e(\overline{S})\} \geq k - 1 = (1/3 - 1/3k)e(F_k)$.

We prove Theorem 1.1 by way of contradiction. Suppose G is a counterexample to Theorem 1.1 and choose G so that $|V(G)|$ is minimum. Results in [19] allow us to show that $\delta(G) \leq 4$ and $e(G) \leq 3(|V(G)| - |M| - 1)$ (and $e(G) \leq 3(|V(G)| - |M| - 1 - \delta(G))$ when $|V(G)|$ is even), where M is any symmetric matching in G; see Section 2) where we also show that $|V(G)| \geq 7$.

We then prove lower bounds on $e(G)$ and $|V(G)|$. This is done by studying the distribution of vertices of small degree in G, using structural arguments and the discharging method. Let $V_i(G)$ denote the vertices of G with degree i. In Section 3, we study properties of G related to $V_2(G)$ (for example, if $|V_2(G)| \geq 5$ then $V_i(G) = \emptyset$ for $4 \leq i \leq 8$), and show that $|V(G)|$ is even. In Section 4, we study the properties of G related to vertices in $V_3(G)$. In Section 5, we derive bounds on $|V_2(G)$ and $|V_3(G)|$. In particular, we show $|V_2(G)\cup V_3(G)|\ge 2$, $|V_3(G)|\le 3$, and if $V_2(G)\ne\emptyset$ then $|V_2(G)|\ge 4(|V(G)|-1)/7$. Using results from Sections 2, 3 and 4, we show $|V(G)|\ge 6$, $e(G)\ge 3((20|V(G)|-21)/29$, and $e(G)\ge 3(4|V(G)|-3)/5$ when $\delta(G)=3$ (where we use discharging method).

To deal with the remaining case (when $e(G)$ is not too big and not too small), we use an idea from [13] about "free vertices" with respect to a maximum matching. We first derive a bound on the size of a maximum bisection in G, which is done in Section 6. We complete the proof of Theorem 1.1 in Section 7 where we show that by deleting at most two edges from G we obtain a complete bipartite graph with one color class consisting of two vertices.

We conclude this introductory section with notation used frequently in the remainder of the paper. Let G be a graph. Let X and Y be two disjoint subsets of $V(G)$, we use $e(X,Y)$ to denote the number of edges with one end in X and the other in Y. We use $G-X$ to denote the graph obtained from G by deleting X and all edges of G incident with X. For two non-adjacent vertices x,y of G, we use $G+xy$ to denote the graph obtained from G by adding the edge xy. For $i\in\{\delta(G),\dots,\Delta(G)\}$, let $V_i(G)$ (or simply V_i if there is no danger of confusion) denote the set of vertices of degree i in G. For any $x\in V(G)$, we use $N_G(x)$ to denote the neighborhood of x. Again, we drop the reference to G when there is no danger of confusion.

2 Bounds for a minimum counterexample

In this section, we present a few useful facts about a possible counterexample to Theorem 1.1. First, note that if G is a counterexample to Theorem 1.1 then G is not a triangle and, for any bisection $[S,\overline{S}]$ of G, $\max\{e(S),e(\overline{S})\}\ge e(G)/3$. We need the following observation about any counterexample to Theorem 1.1.

Lemma 2.1. *Let G be a counterexample to Theorem 1.1. Then G is not complete, and $|V(G)|\ge 7$. Moreover, if G is a counterexample with $|V(G)|$ minimum then G is connected.*

Proof. Clearly, $|V(G)|\ge 4$ as otherwise G is a triangle. If $|V(G)|=4$ then $e(G)\ge 4$; so for any bisection $[S,\overline{S}]$ of G, $\max\{e(S),e(\overline{S})\}\le 1<e(G)/3$, a contradiction. So $|V(G)|\ge 5$.

If G is complete then for any bisection $[S,\overline{S}]$ of G,

$$\max\{e(S),e(\overline{S})\}=\binom{\lceil|V(G)|\rceil/2}{2}<\binom{|V(G)|}{2}/3=e(G)/3,$$

a contradiction. So G is not complete.

Now suppose $|V(G)|=5$. If there exists $u\in V_2$ then let $S=\{u,v,w\}$ such that $uv,uw\notin E(G)$; now $[S,\overline{S}]$ is a bisection of G such that $\max\{e(S),e(\overline{S})\}\le 1<e(G)/3$, a contradiction. So $\delta(G)\ge 3$, and hence $e(G)\ge 8$. Let $S=\{u,v,w\}$ such that $uv\notin E(G)$. Then $[S,\overline{S}]$ is a bisection of G such that $\max\{e(S),e(\overline{S})\}\le 2<e(G)/3$, a contradiction.

We claim $|V(G)|\ge 7$. For, otherwise, $|V(G)|=6$. Then $e(G)\le 9$ as otherwise, for any bisection $[S,\overline{S}]$ of G, $\max\{e(S),e(\overline{S})\}\le 3<e(G)/3$, a contradiction. Also, $e(G)\ge 7$; for otherwise, G is a cycle of length 6 or consists of two disjoint triangles; in each case, G has a

bisection $[S, \overline{S}]$ such that $\max\{e(S), e(\overline{S})\} \leq 1 < e(G)/3$, a contradiction. Let $u, v \in V(G)$ such that $uv \notin E(G)$. Then $G - \{u, v\} \neq K_4$; for otherwise $e(G) \geq 10$ (as $\delta(G) \geq 2$), a contradiction. So let $x, y \in V(G) \setminus \{u, v\}$ with $xy \notin E(G)$, and let $[S, \overline{S}]$ be a bisection of G such that $\{u, v\} \subseteq S$ and $\{x, y\} \subseteq \overline{S}$. Now $\max\{e(S), e(\overline{S})\} \leq 2 < e(G)/3$, a contradiction.

Finally assume that G is a counterexample with $|V(G)|$ minimum, and suppose G is not connected. Then there exist vertex disjoint subgraphs G_1, G_2 of G such that $G = G_1 \cup G_2$. Note that $\delta(G_i) \geq 2$, and each G_i has a bisection $[S_i, \overline{S_i}]$ such that $\max\{e(S_i), e(\overline{S_i})\} < e(G_i)/3$ unless G_i is a triangle. Without loss of generality, assume $|S_1| \leq |\overline{S_1}|$ and $|S_2| \geq |\overline{S_2}|$. Thus, with $S = S_1 \cup S_2$, $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} < (e(G_1) + e(G_2))/3 = e(G)/3$ (when one of G_i is not a triangle), or $\max\{e(S), e(\overline{S})\} = 1 < e(G)/3$ when both G_i are triangles. In either case, we obtain a contradiction. ∎

Next, we provide bounds on the maximum degree and size of counterexamples to Theorem 1.1. They are consequences of a result from [19] on the Bollobás–Scott conjecture. A matching M in a graph G is said to be *symmetric* if each edge in M joins two vertices of the same degree in G.

Lemma 2.2. (*Xu, Yan and Yu*) *Let G be a graph, and let M be a symmetric matching in G. Then G admits a bisection $[S, \overline{S}]$ such that $e(S, \overline{S}) \geq (e(G) + |M|)/2$ and*

(1) $\max\{e(S), e(\overline{S})\} \leq (e(G) - |M| + \Delta(G) - \delta(G))/4$ *if $|V(G)|$ is even,*

(2) $\max\{e(S), e(\overline{S})\} \leq (e(G) - |M| + \Delta(G))/4$ *if $|V(G)|$ is odd.*

In [19], Xu, Yan and Yu show that if G is a graph such that $\delta(G) \geq 5$ or $e(G) \geq 3|V(G)|$, then G admits a good bisection. Here we use the same proof to give a stronger version.

Lemma 2.3. *Let G be a graph. If $\delta(G) \geq 5$ or $e(G) > 3(|V(G)| - 1)$, then G admits a bisection $[S, \overline{S}]$ such that $\max\{e(S), e(\overline{S})\} < e(G)/3$.*

Proof. By Lemma 2.2, let $[S, \overline{S}]$ be a bisection of G such that $\max\{e(S), e(\overline{S})\} \leq (e(G) - |M| + \Delta(G))/4$, where M is a symmetric matching in G.

We may assume $\Delta \geq e(G)/3$; otherwise $\max\{e(S), e(\overline{S})\} < (e(G) + e(G)/3)/4 = e(G)/3$. In particular, we are done if $e(G) > 3(|V(G)| - 1)$ as then $\Delta(G) \leq |V(G)| - 1 < e(G)/3$. Thus we may assume $e(G) \leq 3(|V(G)| - 1)$.

Now assume $\delta(G) \geq 5$ and let t denote the number of vertices of degree 5 in G. Thus $e(G) \geq 5|V(G)|/2 \geq 35/2$ (by Lemma 2.1); so $\Delta(G) \geq e(G)/3 \geq 6$ and $t \leq |V(G)| - 1$. Hence

$$2e(G) \geq \Delta(G) + 5t + 6(|V(G)| - t - 1) \geq e(G)/3 + 6|V(G)| - t - 6 \geq e(G)/3 + 5(|V(G)| - 1);$$

so $e(G) \geq 3(|V(G)| - 1)$. However, $e(G) \leq 3(|V(G)| - 1)$; so $e(G) = 3(|V(G)| - 1)$, $t = |V(G)| - 1$, and $\Delta(G) = e(G)/3$. Therefore, we may choose M so that $|M| \geq 1$. Now $\max\{e(S), e(\overline{S})\} \leq (e(G) - 1 + e(G)/3)/4 < e(G)/3$. ∎

Corollary 2.4. *Let G be a counterexample to Theorem 1.1, and let M be any symmetric matching in G. Then $\delta(G) \leq 4$ and*

(1) *if $|V(G)|$ is odd then $\Delta(G) \geq e(G)/3 + |M|$ and $e(G) \leq 3(|V(G)| - |M| - 1)$,*

(2) *if* $|V(G)|$ *is even then* $\Delta(G) \geq e(G)/3+|M|+\delta(G)$ *and* $e(G) \leq 3(|V(G)|-|M|-\delta(G)-1)$.

Proof. By Lemma 2.3, $\delta(G) \leq 4$. Suppose $|V(G)|$ is odd. By Lemma 2.2, $(e(G) + \Delta(G) - |M|)/4 \geq e(G)/3$, which implies $\Delta(G) \geq e(G)/3 + |M|$. Thus, since $\Delta(G) \leq |V(G)| - 1$, $e(G) \leq 3(|V(G)| - |M| - 1)$. Now assume $|V(G)|$ is even. Then by Lemma 2.2, $(e(G) - |M| + \Delta(G) - \delta(G))/4 \geq e(G)/3$. So $\Delta(G) \geq e(G)/3 + |M| + \delta(G)$. Again, since $\Delta(G) \leq |V(G)| - 1$, $e(G) \leq 3(|V(G)| - |M| - \delta(G) - 1)$. ∎

3 Vertices of degree 2

Suppose G is a counterexample to Theorem 1.1, with $|V(G)|$ minimum. For convenience, let $m = e(G)$ and $n = |V(G)|$. Thus, G is not a triangle and for any bisection $[S, \overline{S}]$ of G, $\max\{e(S), e(\overline{S})\} \geq m/3$. We will prove

- $e(V_2, V_i) = 0$ for $2 \leq i \leq 6$,
- $V_4 \cup V_5 = \emptyset$ if $|V_2| \geq 1$,
- $V_6 \cup V_7 = \emptyset$ if $|V_2| \geq 3$, and
- $V_8 = \emptyset$ if $|V_2| \geq 5$.

As a consequence, we will also show that n is even. This will be useful as we can see from Corollary 2.4 that when n is even the bounds we have for $\Delta(G)$ and $e(G)$ are better.

Lemma 3.1. $e(V_2) = 0$.

Proof. Suppose $e(V_2) \geq 1$, and let $P = x_1x_2 \dots x_{i-1}$, $i \geq 3$, be a maximal path in G such that $V(P) \subseteq V_2$. Let $x_0 \in N(x_1) \setminus \{x_2\}$ and $x_i \in N(x_{i-1}) \setminus \{x_{i-2}\}$, and when possible we further choose P so that $x_0 \neq x_i$.

We claim that $i = 3$, or $i = 4$ and $x_0 = x_4$. For, otherwise, let $H' = G - \{x_1, x_2\} + x_0x_3$. Then H' is not a triangle (as $|V(G)| \geq 7$ by Lemma 2.1), $\delta(H') \geq 2$, and $e(H') = m - 2$. Thus, H' has a good bisection $[T, \overline{T}]$. By symmetry, assume $x_0 \in \overline{T}$. Let $S = T \cup \{x_1\}$. Since $x_0x_3 \in E(H') \setminus E(G)$, $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} = \max\{e(T), e(\overline{T})\} \leq (m-2)/3 < m/3$, a contradiction.

Let $H = G - \{x_1, \dots, x_{i-1}\}$. Then $e(H) = m - i$, and H is not a triangle as $|V(G)| \geq 7$ by Lemma 2.1.

Case 1. $\delta(H) < 2$.

Thus $x_0 = x_i$ and $d(x_0) \leq 3$. By Lemma 2.1, $d(x_0) = 3$. Let $w \in N(x_0) \setminus \{x_1, x_{i-1}\}$.

Suppose $d(w) \geq 3$. Then $\delta(H - x_0) \geq 2$ and $e(H - x_0) = m - i - 1$; and hence $H - x_0$ has a good bisection $[T, \overline{T}]$. By symmetry, we may assume $w \in \overline{T}$. Let $S = T \cup \{x_0\}$ (when $i = 3$ and $|T| \geq |\overline{T}|$) or $S = T \cup \{x_0, x_2\}$ (when $i = 4$ or $|T| < |\overline{T}|$); then $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + 1 \leq (m-i-1)/3 + 1 < m/3$, a contradiction.

Thus $d(w) = 2$, and let $z \in N(w) \setminus \{x_0\}$. Now $d(z) \geq 3$; otherwise we may choose P with $x_0 \neq x_i$. So $\delta(H - \{w, x_0\}) \geq 2$ and, hence, $H - \{w, x_0\}$ admits a good bisection $[T, \overline{T}]$. By symmetry, we may assume $z \in \overline{T}$. If $i = 3$ let $S = T \cup \{w, x_1\}$; and if $i = 4$ let $S = T \cup \{x_0, x_2\}$ (when $|T| > |\overline{T}|$) or $S = T \cup \{w, x_1, x_3\}$ (when $|T| \le |\overline{T}|$). Then $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \le \max\{e(T), e(\overline{T})\} + 1 \le (m - i - 2)/3 + 1 < m/3$, a contradiction.

Case 2. $\delta(H) \ge 2$.

Then by the choice of G, H has a bisection $[T, \overline{T}]$ such that $\max\{e(T), e(\overline{T})\} < (m - i)/3$. By symmetry we may assume $x_0 \in \overline{T}$.

If $i = 3$ then, with $S = T \cup \{x_1\}$, $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \le \max\{e(T), e(\overline{T})\} + 1 < m/3$, a contradiction. Thus $i = 4$ and $x_0 = x_4$. If $|T| \le |\overline{T}|$ then, with $S = T \cup \{x_1, x_3\}$, $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \le \max\{e(T), e(\overline{T})\} + 1 < m/3$, a contradiction. So $|T| > |\overline{T}|$, and hence $|V(G)|$ is even.

Let $w \in N(x_0) \cap V(H)$, and $H' = G - \{x_2, x_3\} - x_0w + x_1w$. Then $|V(H')|$ is even (so H' is not a triangle), $\delta(H') \ge 2$, and $e(H') = m - 3$. By the choice of G, H' has a bisection $[R, \overline{R}]$ such that $\max\{e(R), e(\overline{R})\} < (m - 3)/3$. By symmetry we may assume $x_0 \in \overline{R}$. Let $S = R \cup \{x_3\}$. Then $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \le \max\{e(R), e(\overline{R})\} + 1 < m/3$, a contradiction. ∎

In order to prove $e(V_2, V_i) = \emptyset$ for $3 \le i \le 6$ and other results, we need the following result about the neighborhood of two vertices in V_2.

Lemma 3.2. *$G[N(x) \cup N(y)]$ is complete for distinct $x, y \in V_2$.*

Proof. Let $x, y \in V_2$ be distinct, and assume there exist distinct $u, v \in N(x) \cup N(y)$ such that $uv \notin E(G)$. We choose u, v so that, when possible, $\{u, v\} \subseteq N(x)$ or $\{u, v\} \subseteq N(y)$. Then by Lemma 3.1, $H := G - \{x, y\} + uv$ has minimum degree at least 2. Clearly, $e(H) = m - 3$. By Lemma 2.1, $|V(H)| \ge 4$; so H is not a triangle. Hence, H has a bisection $[T, \overline{T}]$ such that $\max\{e(T), e(\overline{T})\} < (m - 3)/3$.

By symmetry, we may assume $|N(x) \cap \overline{T}| \ge |N(x) \cap T|$; so $e(x, T) \le 1$. Let $S = T \cup \{x\}$; then $e(S) \le e(T) + 1 < m/3$ and $[S, \overline{S}]$ is a bisection of G. If $N(y) \not\subseteq \overline{T}$ or $\{u, v\} \subseteq \overline{T}$ then $e(\overline{S}) \le e(\overline{T}) + 1 < m/3$, a contradiction. So $N(y) \subseteq \overline{T}$ and $\{u, v\} \not\subseteq \overline{T}$; hence $e(x, T) = 1$. Then, with $R = T \cup \{y\}$, $[R, \overline{R}]$ is a bisection of G such that $\max\{e(R), e(\overline{R})\} = \max\{e(T), e(\overline{T})\} + 1 < m/3$, a contradiction. ∎

Lemma 3.3. *$e(V_2, V_3) = 0$.*

Proof. Suppose there exists $uv \in E(G)$ with $u \in V_3$ and $v \in V_2$, and let $N(u) = \{v, x, y\}$.

Suppose $x, y \notin V_2$. By symmetry, assume $vx \in E(G)$ if $\{x, y\} \cap N(v) \ne \emptyset$. Let $H = G - \{u, v\}$ if $\{x, y\} \cap N(v) = \emptyset$ or $d(x) \ge 4$; otherwise let $z \in V(G) \setminus (N(x) \cup \{x\})$ (which exists as $|V(G)| \ge 7$ by Lemma 2.1) and $H = G - \{u, v\} + xz$. Then $\delta(H) \ge 2$ and $e(H) \le m - 3$. By Lemma 2.1, H is not a triangle. Thus, H admits a bisection $[T, \overline{T}]$ such that $\max\{e(T), e(\overline{T})\} < (m - 3)/3$. By symmetry assume $e(u, T) \le 1$. Then, with $S = T \cup \{u\}$, $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \le \max\{e(T), e(\overline{T})\} + 1 < m/3$, a contradiction.

Hence, we may assume $x \in V_2$. Then $y \in N(v) \cap N(x)$ by Lemma 3.2. By Lemma 2.1, there exists $w \in N(y) \setminus \{u, v, x\}$, and let $H = G - \{u, v\} + wx$. Then $\delta(H) \ge 2$, $e(H) = m - 3$, and H is not a triangle (by Lemma 2.1). So H has a bisection $[T, \overline{T}]$ such that $\max\{e(T), e(\overline{T})\} <$

$(m-3)/3$. By symmetry assume $y \in T$. Let $S = T \cup \{v\}$. Then $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + 1 < m/3$, a contradiction. ∎

We need two results about vertices adjacent to V_2, in order to show that such vertices have reasonably large degree.

Lemma 3.4. *For any $u_1u_2 \in E(G)$ with $|N(u_1) \cap N(u_2) \cap V_2| \geq 2$, $\min\{d(u_1), d(u_2)\} \geq |N(u_1) \cap N(u_2) \cap V_2| + 3$.*

Proof. Suppose there exists $u_1u_2 \in E(G)$ such that $N(u_1) \cap N(u_2) \cap V_2 = \{v_1, \ldots, v_h\}$, $h \geq 2$, $d(u_1) \leq d(u_2)$, and $d(u_1) \leq h+2$. Then $d(u_2) \geq h+2$; otherwise $G - u_1u_2 \cong K_{2,h}$ and, with $S = \{u_1, v_1, \ldots, v_{\lfloor h/2 \rfloor}\}$, $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \leq (h+1)/2 < m/3$, a contradiction. Let $w_2 \in N(u_2) \setminus \{u_1, v_1, \ldots, v_h\}$, and if $d(u_1) = h+2$ let $w_1 \in N(u_1) \setminus \{u_2, v_1, \ldots, v_h\}$. By Lemma 3.2 and by the definition of h, $d(w_2) \geq 3$ and, if w_1 is defined, $d(w_1) \geq 3$.

We claim that $d(u_2) = h+2$. For, suppose $d(u_2) \geq h+3$, and let $H = G - \{u_1, v_1, \ldots, v_h\}$. Then $\delta(H) \geq 2$, $e(H) \leq m - 2h - 1$, and $e(H) \leq m - 2h - 2$ when $d(u_1) = h+2$. By the choice of G, H has a good bisection $[T, \overline{T}]$, and we choose $[T, \overline{T}]$ so that $\max\{e(T), e(\overline{T})\}$ is minimum; then $\max\{e(T), e(\overline{T})\} < e(H)/3$ unless H is a triangle. By symmetry, assume $u_2 \in T$. First, assume $|\overline{T}| < |T|$. Let $S = T \cup \{v_1, \ldots, v_{\lceil h/2 \rceil}\}$. Then $[S, \overline{S}]$ is a bisection of G, and $e(S) \leq e(T) + h/2 < m/3$. If $d(u_1) = h+1$ or $w_1 \in T$ then $e(\overline{S}) \leq e(\overline{T}) + h/2 < m/3$, a contradiction. So $d(u_1) = h+2$ and $w_1 \in \overline{T}$. Then H is not a triangle, as otherwise we may choose $[T, \overline{T}]$ with $u_2, w_1 \in T$, and we would obtain a contradiction as above. Hence $\max\{e(T), e(\overline{T})\} < (m-2h-2)/3$; so $e(\overline{S}) \leq e(\overline{T}) + h/2 + 1 < (m-2h-2)/3 + h/2 + 1 \leq m/3$, a contradiction. Now assume $|T| \leq |\overline{T}|$. Let $S = T \cup \{v_1, v_2, \ldots, v_{\lceil (h+1)/2 \rceil}\}$; then $[S, \overline{S}]$ is a bisection of G such that

$$\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + \lceil (h+1)/2 \rceil.$$

When $h \notin \{2, 4\}$, we have $\max\{e(S), e(\overline{S})\} \leq (m-2h-1)/3 + \lceil (h+1)/2 \rceil < m/3$, a contradiction. When $d(u_1) = h+2$, $\max\{e(S), e(\overline{S})\} \leq (m-2h-2)/3 + (h+2)/2$, which is less than $m/3$ unless $h = 2$. Suppose $h = 2$. Then H is not a triangle by Lemma 2.1 and $d(u_1) = h+2$ by Lemma 3.3; so $\max\{e(S), e(\overline{S})\} < (m-2h-2)/3 + (h+2)/2 = m/3$, a contradiction. Hence $h = 4$. Then $d(u_1) = h+1$, and $\max\{e(S), e(\overline{S})\} \leq (m-2h-1)/3 + (h+2)/2 = m/3$; so equality must hold by the choice of G. Therefore, H must be a triangle. Clearly, $|V(G)| = 8$ and $e(G) = 12$. With $S = \{u_2, v_1, v_2, v_3\}$, $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} = 3 < e(G)/3$, a contradiction.

Suppose $w_2 \in N(u_1)$ and $w_2 \in V_3$. Then $w_1 = w_2$. Let $w \in N(w_1) \setminus \{u_1, u_2\}$; so $d(w) \geq 3$ by Lemma 3.3. Let $H = G - u_1 - N(u_1)$. Then $\delta(H) \geq 2$ and $e(H) = m - 2h - 4$. By the choice of G, H has a good bisection $[T, \overline{T}]$ and, by symmetry, assume $w \in T$. If $|T| \leq |\overline{T}|$ let $S = T \cup \{u_1, v_1, \ldots, v_{\lceil (h+1)/2 \rceil}\}$, and otherwise let $S = T \cup \{u_1, v_1, \ldots, v_{\lceil h/2 \rceil}\}$. Now $[S, \overline{S}]$ is a bisection of G such that

$$\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + (h+2)/2 \leq (m-2h-4)/3 + (h+2)/2 < m/3,$$

a contradiction.

So $w_2 \notin N(u_1)$ or $w_2 \notin V_3$. Let $H = G - u_2 - (N(u_2) \setminus \{w_2\})$; then $\delta(H) \geq 2$, $e(H) \leq m - 2h - 2$, and $e(H) = m - 2h - 3$ if w_1 is defined. Suppose H is a triangle. Then let $S = \{u_1, w_2, v_1, \ldots, v_{\lceil h/2 \rceil}\}$. Now $[S, \overline{S}]$ is a bisection of G. Since $m \geq 2h + 4$, $\max\{e(S), e(\overline{S})\} \leq (h+1)/2 + 1 < m/3$, a contradiction. Thus H is not a triangle. Hence, H has a bisection $[T, \overline{T}]$ such that $\max\{e(T), e(\overline{T})\} < e(H)/3$ and, by symmetry, assume $w_2 \in \overline{T}$.

If $|T| \leq |\overline{T}|$ let $S = T \cup \{u_2, v_1, v_2, \ldots, v_{\lceil h/2 \rceil}\}$; then $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + (h+2)/2 < (m - 2h - 2)/3 + (h+2)/2 \leq m/3$, a contradiction. So $|T| > |\overline{T}|$, and let $S = T \cup \{u_2, v_1, \ldots, v_{\lfloor h/2 \rfloor}\}$. Then $e(S) \leq e(T) + (h+2)/2 < (m - 2h - 2)/3 + (h+2)/2 = m/3$. If w_1 is not defined then $e(\overline{S}) \leq e(T) + (h+1)/2 < (m - 2h - 2)/3 + (h+1)/2 < m/3$, a contradiction. So w_1 is defined. If h is even then $e(\overline{S}) \leq e(T) + (h+2)/2 < (m - 2h - 3)/3 + (h+2)/2 < m/3$, a contradiction. Then h is odd; so $h \geq 3$. Then $e(\overline{S}) \leq e(T) + (h+1)/2 + 1 < (m - 2h - 3)/3 + (h+3)/2 \leq m/3$, a contradiction. ∎

Lemma 3.5. *For $x \in V_i$ and $i \leq 6$, $|N(x) \cap V_2| \neq 1$.*

Proof. Assume that $x \in V_i$, $i \leq 6$ and $N(x) \cap V_2 = \{v\}$. Let $H = G - \{v, x\}$. Then $e(H) = m - i - 1$, $i \in \{4, 5, 6\}$ and $\delta(H) \geq 2$ (by Lemmas 3.1 and 3.3), and $H \neq K_3$ (by Lemma 2.1). So H has a bisection $[T, \overline{T}]$ such that $\max\{e(T), e(\overline{T})\} < e(H)/3$. By symmetry assume $|N(x) \cap T| \leq \lfloor (i-1)/2) \rfloor$. Let $S = T \cup \{x\}$. Then, since $i \leq 6$,

$$\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + \lfloor (i-1)/2 \rfloor < (m - i - 1)/3 + \lfloor (i-1)/2 \rfloor \leq m/3,$$

a contradiction. ∎

The following result immediately implies that $e(V_2, V_i) = \emptyset$ for $i = 4, 5$.

Lemma 3.6. *If $V_2 \neq \emptyset$ then $V_4 \cup V_5 = \emptyset$.*

Proof. Suppose $V_2 \neq \emptyset$ and $V_4 \cup V_5 \neq \emptyset$. Let $u \in V_4 \cup V_5$ and $v \in V_2$ such that $uv \in E(G)$ if $N(u) \cap V_2 \neq \emptyset$, and let $H = G - \{u, v\}$. Then H is not a triangle by Lemma 2.1.

We claim that $uv \in E(G)$. For, suppose $uv \notin E(G)$. Then by the choice of u and v, $N(u) \cap V_2 = \emptyset$. Thus, $e(H) \leq m - 6$ and $\delta(H) \geq 2$ (by Lemmas 3.1 and 3.3). So H has a bisection $[T, \overline{T}]$ such that $\max\{e(T), e(\overline{T})\} < e(H)/3$. By symmetry assume $|N(u) \cap \overline{T}| \leq 2$, and let $S = T \cup \{v\}$. Then $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + 2 < e(H)/3 + 2 \leq m/3$, a contradiction.

Now suppose $u \in V_4$ and let $N(u) = \{v_1, v_2, v_3, v_4\}$, with $v_1 = v$. By Lemma 3.5, let $v_2 \in V_2$. So by Lemmas 3.1 and 3.2, $N(v_i) \subseteq \{u, v_3, v_4\}$ for $i = 1, 2$. If $N(v_1) = N(v_2) = \{u, v_s\}$ for some $s \in \{3, 4\}$, then the edge uv_s contradicts Lemma 3.4. So we may assume $v_1v_3, v_2v_4 \in E(G)$. Then $v_3v_4 \in E(G)$ by Lemma 3.2, and $v_3, v_4 \notin V_3$ by Lemma 3.3. Let $H = G - \{u, v_1, v_2\}$. Then $e(H) = m - 6$ and $\delta(H) \geq 2$. Note that H is not a triangle by Lemma 2.1. So H has a bisection $[T, \overline{T}]$ such that $\max\{e(T), e(\overline{T})\} < e(H)/3$. By symmetry assume $|T| \leq |\overline{T}|$, and let $S = T \cup \{v_1, v_2\}$. Then $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + 2 < e(H)/3 + 2 \leq m/3$, a contradiction.

Thus $u \in V_5$, and let $N(u) = \{v_1, v_2, v_3, v_4, v_5\}$, with $v_1 = v$. By Lemma 3.5, let $v_2 \in V_2$. By Lemmas 3.1 and 3.2, $N(v_i) \subseteq \{u, v_3, v_4, v_5\}$ for $i = 1, 2$.

First, assume $N(x) \neq N(y)$ for any distinct $x, y \in N(u) \cap V_2$. Without loss of generality, assume $v_1v_3, v_2v_4 \in E(G)$. Then by Lemma 3.2, $v_3v_4 \in E(G)$ and $d(v_5) \geq 3$ (as otherwise $N(v_5) = N(v_i)$ for some $i \in \{1, 2\}$). Let $H = G - \{u, v_1, v_2\}$; then $e(H) = m - 7$. By Lemmas 3.3, $d(v_3) \geq 4$ and $d(v_4) \geq 4$. So $\delta(H) \geq 2$. By the choice of G, let $[T, \overline{T}]$ be a good bisection of H and, by symmetry, assume $v_5 \in \overline{T}$. If $|T| > |\overline{T}|$ let $S = T \cup \{u\}$; and otherwise let $S = T \cup \{u, v_1\}$ (when $\{v_3, v_4\} \subseteq \overline{T}$) or $S = T \cup \{v_1, v_2\}$ (when $\{v_3, v_4\} \not\subseteq \overline{T}$). Then $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + 2 \leq e(H)/3 + 2 < m/3$, a contradiction.

Thus we may assume $N(v_1) = N(v_2) = \{u, v_5\}$. Then $d(v_5) \geq 5$ by Lemma 3.4.

Suppose $d(v_3) \geq 3$ and $d(v_4) \geq 3$. Let $H = G - \{u, v_1, v_2\}$. Then $e(H) = m - 7$ and $\delta(H) \geq 2$. So H has a good bisection $[T, \overline{T}]$ and, by symmetry, assume $v_5 \in T$. If $|T| \leq |\overline{T}|$ then let $S = T \cup \{v_1, v_2\}$; otherwise, let $S = T \cup \{u\}$ (when $\{v_3, v_4\} \subseteq \overline{T}$), and $S = T \cup \{v_1\}$ (when $\{v_3, v_4\} \not\subseteq \overline{T}$). Then $\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + 2 \leq e(H)/3 + 2 < m/3$, a contradiction.

So assume $d(v_3) = 2$. By Lemma 3.4, $v_3v_5 \notin E(G)$; so $v_3v_4, v_4v_5 \in E(G)$ by Lemma 3.2, and $d(v_4) \geq 4$ by Lemma 3.3. Let $H = G - \{u, v_1, v_2, v_3\}$. Then $e(H) = m - 8$ and $\delta(H) \geq 2$. So H has a good bisection $[T, \overline{T}]$ and, by symmetry, assume $v_5 \in T$. If $v_4 \in \overline{T}$ let $S = T \cup \{u, v_3\}$; and otherwise let $S = T \cup \{v_1, v_2\}$. Then $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + 2 \leq e(H)/3 + 2 < m/3$, a contradiction. ∎

In order to study V_k for $6 \leq k \leq 9$, we need to bound $|N(u) \cap V_2|$ for $u \in V_k$.

Lemma 3.7. *Let $u \in V_k$ and $k \geq 6$. Then*

(1) $|N(u) \cap V_2| < (4k-6)/5$ *if* $|N(u) \cap V_2|$ *is even, and* $|N(u) \cap V_2| < (4k-9)/5$ *if* $|N(u) \cap V_2|$ *is odd,*

(2) $|N(u) \cap V_2| < (2k-4)/(1+5p)$ *if* $0 < p \leq 1$ *and* $|V_2 \setminus N(u)| \geq p|N(u) \cap V_2|$*, and*

(3) $|N(u) \cap V_2| < (k-2)/(p+2)$ *if* $p \geq 2$ *is an integer,* $|N(u) \cap V_2| \leq (k-3)/(p-1)$*, and at least* $p|N(u) \cap V_2|$ *elements of* $V_2 \setminus N(u)$ *have the same neighborhood in* G.

Proof. Let $N(u) \cap V_2 = \{v_1, \ldots, v_l\}$. We may assume $l \geq 1$ as otherwise (1)–(3) all hold. We may also assume $|V_2| \geq 2$, for otherwise, (1) holds, and the assumptions of (2) and (3) would not hold. Thus, by Lemmas 3.1 and 3.2, $N(V_2) \subseteq N(u) \setminus \{v_1, \ldots, v_l\}$.

To prove (1), let $H = G - \{u, v_1, \ldots, v_l\}$. Note that $e(H) = m - k - l$. If $d_H(x) \leq 1$ for some $x \in V(H)$ then $x \in N(\{v_1, \ldots, v_l\})$ and hence $d_G(x) \geq 6$ (by Lemmas 3.1, 3.3 and 3.6); so $|N(x) \cap \{v_1, \ldots, v_l\}| \geq 2$ and, hence, by Lemma 3.2, $xu \in E(G)$ which contradicts Lemma 3.4. Hence $\delta(H) \geq 2$. So H has a good bisection $[T, \overline{T}]$ such that $\max\{e(T), e(\overline{T})\} < e(H)/3$ unless H is a triangle. By symmetry assume $|N(T) \cap \{v_1, \ldots, v_l\}| \geq |N(\overline{T}) \cap \{v_1, \ldots, v_l\}|$ and, if H is a triangle, we may choose T so that $|\overline{T}| < |T|$. Thus, $e(\overline{T}) < e(H)/3$. Without loss of generality, let $N(v_i) \subseteq T \cup \{u\}$ for $i \leq h = \lfloor l/2 \rfloor$. Let $S = T \cup \{v_{h+1}, \ldots, v_l\}$ if l is odd or $|\overline{T}| \leq |T|$, and $S = T \cup \{v_h, \ldots, v_l\}$ otherwise. If H is not a triangle then

$$e(S) \leq e(T) + \lceil (l+1)/2 \rceil < (m-k-l)/3 + (l+2)/2 \leq m/3,$$

and if H is a triangle then

$$e(S) \leq e(T) + \lceil l/2 \rceil \leq (m-k-l)/3 + (l+1)/2 < m/3.$$

Since (1) holds if $l = 1$, we may assume $l \geq 2$. Then $|N(u) \cap T| \geq 1$ by Lemma 3.2. Thus, if l is even then

$$m/3 \leq e(\overline{S}) \leq e(\overline{T}) + l/2 + k - l - 1 < (m-k-l)/3 + l/2 + k - l - 1 = m/3 - (5l - 4k + 6)/6;$$

so $l < (4k-6)/5$. If l is odd then

$$m/3 \leq e(\overline{S}) \leq e(\overline{T}) + (l-1)/2 + k - l - 1 < (m-k-l)/3 + (l-1)/2 + k - l - 1 = m/3 - (5l - 4k + 9)/6;$$

so $l < (4k-9)/5$. Thus we have (1).

To prove (2), let $w_1, \ldots, w_h \in V_2 \setminus N(u)$, $h \in \{\lceil pl \rceil, \lceil pl \rceil - 1\}$ with $l + h + 1$ even, and $h' = (l+h+1)/2$. Since $p \leq 1$, $h \leq l$; so $h' \leq l$. Let $H = G - \{u, v_1, \ldots, v_l, w_1, \ldots, w_h\}$. Then $e(H) = m - (k + l + 2h)$. If $\delta(H) \leq 1$ then there exists $v \in N(u) \cap V(H)$ with $|N(v) \cap V(H)| \leq 1$; so $d(v) \geq 6$ by Lemmas 3.1, 3.3 and 3.6, and $|N(v) \cap V_2| \geq d(v) - 2 \geq (4d(v)-6)/5$, contradicting (1). So $\delta(H) \geq 2$ and thus H has a good bisection $[T, \overline{T}]$ such that $\max\{e(T), e(\overline{T})\} < e(H)/3$ unless H is a triangle, and if H is a triangle then we may assume $e(\overline{T}) < e(H)/3$. Let $A_1 = \{v_1, \ldots, v_{h'}\}$ and $A_2 = \{v_{h'+1}, \ldots, v_l, w_1, \ldots, w_h\}$. Then

$$\begin{aligned} & \max\{e(T \cup A_1), e(\overline{T} \cup A_1)\} \\ \leq \ & \max\{e(T), e(\overline{T})\} + h' \\ \leq \ & (m - (k + l + 2h))/3 + (l + h + 1)/2 \\ = \ & m/3 - (2k - l + h - 3)/6 \\ < \ & m/3. \end{aligned}$$

Write $r := |N(u) \cap \overline{T}|$. Then

$$e(\overline{T}, A_2) > (k + l + 2h - 3r)/3 - (l - h');$$

for otherwise, with $S = T \cup A_1$,

$$\begin{aligned} e(\overline{S}) &= e(\overline{T}) + e(u, \overline{T}) + e(u, A_2) + e(\overline{T}, A_2) \\ &< (m - (k + l + 2h))/3 + r + (l - h') + (k + l + 2h - 3r)/3 - (l - h') \\ &= m/3, \end{aligned}$$

and $[S, \overline{S}]$ would be a bisection of G such that $\max\{e(S), e(\overline{S})\} < m/3$, a contradiction. Hence, since $h' = (l + h + 1)/2$,

$$e(T, A_2) = (l - h' + 2h) - e(\overline{T}, A_2) < (l + h - 1) - (k + l + 2h - 3r)/3$$

and

$$
\begin{aligned}
& e(T \cup \{u\} \cup A_2) \\
\leq\ & e(T) + e(u,T) + e(u,A_2) + e(T,A_2) \\
<\ & \frac{m-(k+l+2h)}{3} + (k-l-r) + (l-h') + \left(l+h-1-\frac{k+l+2h-3r}{3}\right) \\
=\ & \frac{m}{3} - \frac{l+5h-2k+9}{6} \\
\leq\ & \frac{m}{3} - \frac{l+5pl-2k+4}{6} \qquad \text{(since } h \geq pl-1\text{)}.
\end{aligned}
$$

Therefore, if $l \geq (2k-4)/(1+5p)$ then $e(T \cup \{u\} \cup A_2) < m/3$; so with $S = T \cup \{u\} \cup A_2$ and $\overline{S} = \overline{T} \cup A_1$, $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} < m/3$, a contradiction. Hence $l < (2k-4)/(1+5p)$, and (2) holds.

We now prove (3). Let $w_1, w_2, \ldots, w_h \in V_2 \backslash N(u)$ such that $N(w_i) = N(w_j)$ for $1 \leq i, j \leq h$, where $h \in \{pl, pl-1\}$ with $h+l+1$ is even. Let $h'' = (h+l-1)/2$. Since $p \geq 2$ and $l \geq 1$, $h \geq l$; hence $h'' < h$.

Let $H = G - \{u, v_1, \ldots, v_l, w_1, \ldots, w_h\}$. Then $e(H) = m - (k+l+2h)$ by Lemma 3.1. Since $l \geq 1$, $|V_2 \setminus N(u)| \geq 2$ and $|V_2| \geq 3$. By the same argument as for (2), $\delta(H) \geq 2$. Hence let $[T, \overline{T}]$ be a good bisection of H such that $\max\{e(T), e(\overline{T})\} < e(H)/3$, unless H is a triangle. By symmetry we may assume $|N(u) \cap \overline{T}| \leq (k-l)/2$ and, moreover, if H is a triangle then $N(w_1) \not\subseteq T$ and $|\overline{T}| = 1$. Thus $e(\overline{T}) < e(H)/3$. Let $A_1 = \{u, w_1, \ldots, w_{h''}\}$, and $A_2 = \{v_1, \ldots, v_l, w_{h''+1}, \ldots, w_h\}$. Assume for a contradiction that $l \geq (k-2)/(p+2)$.

Suppose $N(w_1) \subseteq \overline{T}$. Then $N(w_i) \subseteq N(u) \cap \overline{T}$ for $1 \leq i \leq h$, and $k - l \geq 4$. Hence H is not a triangle, and so $\max\{e(T), e(\overline{T})\} < e(H)/3$. Let $S = T \cup A_1$. Then, since $h \geq pl-1$ and $l \geq (k-2)/(p+2)$,

$$
e(S) \leq e(T) + (k-l-2) < \frac{m-(k+l+2h)}{3} + (k-l-2) \leq \frac{m}{3} - \frac{(4+2p)l-2k+4}{3} \leq \frac{m}{3},
$$

and, since $l \leq (k-3)/(p-1)$ (by assumption in (3)) and $h \leq pl$,

$$
e(\overline{S}) \leq e(\overline{T}) + l + 2(h-h'') < \frac{m-(k+l+2h)}{3} + h + 1 = \frac{m}{3} - \frac{k+l-h-3}{3} \leq \frac{m}{3}.
$$

Hence, $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} < m/3$, a contradiction.

Thus $N(w_1) \not\subseteq \overline{T}$. Let $S = T \cup A_2$. If $N(w_1) \not\subseteq T$ then, since $h \geq pl-1$,

$$
e(S) \leq e(T) + l + (h-h'') \leq \frac{m-(k+l+2h)}{3} + \frac{l+h+1}{2} = \frac{m}{3} - \frac{2k-l+h-3}{6} < \frac{m}{3},
$$

and, since $l \geq (k-2)/(p+2)$ and $h \geq pl-1$,

$$
e(\overline{S}) = e(\overline{T}) + |N(u) \cap \overline{T}| + h'' < \frac{m-(k+l+2h)}{3} + \frac{k-l}{2} + \frac{l+h-1}{2} \leq \frac{m}{3} - \frac{(p+2)l-k+2}{6} \leq \frac{m}{3};
$$

so $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} < m/3$, a contradiction.

Therefore, $N(w_1) \subseteq T$. Then by the choice of $[T, \overline{T}]$, H is not a triangle. Hence $\max\{e(T), e(\overline{T})\} < e(H)/3$. Let $S = T \cup A_2$. Then, since $l \leq h \leq pl$ and $l \leq (k-3)/(p-1)$,

$$\begin{aligned} e(S) &\leq e(T) + l + 2(h - h'') \\ &< \frac{m - (k + l + 2h)}{3} + h + 1 \\ &= \frac{m}{3} - \frac{k + l - h - 3}{3} \\ &\leq \frac{m}{3} - \frac{k + l - pl - 3}{3} \\ &\leq m/3 \end{aligned}$$

and, since $h \geq pl - 1$ and $l \geq (k-2)/(p+2) > (k+4)/(5+4p)$,

$$\begin{aligned} e(\overline{S}) &= e(\overline{T}) + |N(u) \cap \overline{T}| \\ &< \frac{m - (k + l + 2h)}{3} + \frac{k - l}{2} \\ &= \frac{m}{3} - \frac{5l + 4h - k}{6} \\ &\leq \frac{m}{3} - \frac{(5 + 4p)l - (k + 4)}{3} \\ &< \frac{m}{3}, \end{aligned}$$

a contradiction. ∎

We now prove a sufficient condition for $V_k = \emptyset$ when $k \geq 4$.

Lemma 3.8. *Let $k \geq 4$ and assume that k vertices in V_2 have the same neighborhood in G. Then $V_i = \emptyset$ for $4 \leq i \leq k$.*

Proof. Let $v_1, v_2, \ldots, v_k \in V_2$ such that $N(v_i) = \{x, y\}$ for $1 \leq i \leq k$. Then $xy \in E(G)$ by Lemma 3.2, and so $d(x) \geq k + 3$ and $d(y) \geq k + 3$ by Lemma 3.4. Suppose there exists $u \in V_l$ for some $l \in \{4, 5, \ldots, k\}$. Then $l \leq k \leq |V_2 \setminus N(u)|$. By Lemma 3.6, $l \geq 6$. By Lemma 3.7(2) (with $p = 1$), $|N(u) \cap V_2| < (l-2)/3$.

We claim that $N(u) \cap V_2 = \emptyset$. For, suppose $|N(u) \cap V_2| \geq 1$. Then, when $|N(u) \cap V_2| < (l-2)/(p+2)$, we have $l > p + 4$; so $(l-2)/(p+2) < (l-3)/(p+1)$. Hence, as long as $|N(u) \cap V_2| < (l-2)/(p+2)$ we may repeatedly apply Lemma 3.7(3) to u and $\{v_1, \ldots, v_k\}$ (by starting with $p = 3$ and increasing p by 2 at a time), to conclude that $|N(u) \cap V_2| < (l-2)/(p+2)$ for all $p \geq 3$, a contradiction.

Let $h = 2\lfloor l/2 \rfloor - 3$, and $H = G - \{u, v_1, \ldots, v_h\}$. Then h is odd, $h \leq l - 3$, $e(H) = m - l - 2h$, and $\delta(H) \geq 2$ (since $N(u) \cap V_2 = \emptyset$ and $k \geq l$). Moreover, H is not a triangle since $l \geq 6$. Thus H has a bisection $[T, \overline{T}]$ such that $\max\{e(T), e(\overline{T})\} < e(H)/3$.

Suppose $\{x, y\} \subseteq T$. If $|N(u) \cap T| \leq 2$ then let $S = T \cup \{u, v_1, \ldots, v_{(h-1)/2}\}$; so $[S, \overline{S}]$ is a bisection of G, $e(\overline{S}) = e(\overline{T}) < e(H)/3 < m/3$, and

$$e(S) \leq e(T) + (h - 1) + 2 < (m - l - 2h)/3 + h + 1 = m/3 - (l - 3 - h)/3 \leq m/3,$$

a contradiction. Hence $|N(u) \cap T| \geq 3$, and let $S = T \cup \{v_1, \ldots, v_{(h+1)/2}\}$. Then $[S, \overline{S}]$ is a bisection of G, $e(S) = e(T) + h + 1 < m/3$, and

$$e(\overline{S}) \leq e(\overline{T}) + l - |N(u) \cap T| \leq e(\overline{T}) + l - 3 < (m - l - 2h)/3 + l - 3 \leq m/3,$$

a contradiction.

Similarly, we get a contradiction when $\{x, y\} \subseteq \overline{T}$. Thus $|\{x, y\} \cap T| = 1 = |\{x, y\} \cap \overline{T}|$. By symmetry, let $|N(u) \cap T| \leq \lfloor l/2 \rfloor = (h + 3)/2$. Let $S = T \cup \{u, v_1, \ldots, v_{(h-1)/2}\}$. Then $[S, \overline{S}]$ is a bisection of G,

$$e(S) \leq e(T) + (h+3)/2 + (h-1)/2 < (m-l-2h)/3 + (h+1) = m/3 - (l-3-h)/3 \leq m/3,$$

and

$$e(\overline{S}) \leq e(\overline{T}) + (h+1)/2 < (m - l - 2h)/3 + (h+1)/2 < m/3,$$

a contradiction. ∎

For $u \in V_7 \cup V_8$, we show in the next two lemmas that we can bound $|N(u) \cap V_2|$ more precisely.

Lemma 3.9. *$V_2 = \emptyset$, or $|N(u) \cap V_2| = 1$ for all $u \in V_7$.*

Proof. Suppose $V_2 \neq \emptyset$, and let $u \in V_7$ and $N(u) = \{v_1, \ldots, v_7\}$. By Lemma 3.7(1), $|N(u) \cap V_2| \leq 4$.

Suppose $|N(u) \cap V_2| = 0$. Let $v \in V_2$, and let $H = G - \{u, v\}$. Then H is not a triangle, $\delta(H) \geq 2$ (by Lemmas 3.1 and 3.3), and $e(H) = m - 9$. So H has a bisection $[T, \overline{T}]$ such that $\max\{e(T), e(\overline{T})\} < e(H)/3$, and by symmetry assume $|N(u) \cap T| \leq 3$. Let $S = T \cup \{u\}$. Then $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + 3 < (m-9)/3 + 3 = m/3$, a contradiction.

Now assume $|N(u) \cap V_2| \neq 1$. Let $N(u) \cap V_2 = \{v_1, \ldots, v_h\}$, and let $H = G - \{u, v_1, \ldots, v_h\}$. Then $2 \leq h \leq 4$. By Lemma 3.2, $N(V_2) \subseteq \{u, v_{h+1}, \ldots, v_7\}$. So $e(H) = m - 7 - h$. If $\delta(H) \leq 1$ then $|N(v_i) \setminus \{u, v_1, \ldots, v_h\}| \leq 1$ for some $i > h$ and, by Lemmas 3.3 and 3.6, $d(v_i) \geq 6$; so by Lemma 3.7(1), $|N(v_i) \cap V_2| < (4d(v_i) - 6)/5 \leq d(v_i) - 2$, a contradiction. Hence, $\delta(H) \geq 2$. So let $[T, \overline{T}]$ be a good bisection of H.

Suppose $h = 3$. Without loss of generality assume $v_1 v_4 \in E(G)$ and $v_4 \in T$. If $\{v_5, v_6, v_7\} \subseteq \overline{T}$ then let $S = T \cup \{u, v_3\}$, and otherwise let $S = T \cup \{v_2, v_3\}$. Then $[S, \overline{S}]$ is bisection of G and $\max\{e(S), e(\overline{S})\} \leq (m - 10)/3 + 3 < m/3$, a contradiction. Thus, we have two cases to consider: $h = 2$ and $h = 4$.

Case 1. $h = 2$.

In this case, H is not a triangle; so we may choose $[T, \overline{T}]$ such that $\max\{e(T), e(\overline{T})\} < e(H)/3 = (m - 9)/3$.

First, assume there exists $3 \leq i \leq 7$ such that $v_1 v_i, v_2 v_i \in E(G)$, and without loss of generality let $i = 3$ and $v_3 \in T$. If $\{v_4, v_5, v_6, v_7\} \subseteq \overline{T}$ let $S = T \cup \{u\}$ when $|T| \geq |\overline{T}|$ and $S = T \cup \{u, v_1\}$ when $|T| < |\overline{T}|$; then $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + 3 < (m - 9)/3 + 3 = m/3$, a contradiction. If $|\{v_4, v_5, v_6, v_7\} \cap \overline{T}| = 3$

then let $S = T \cup \{u\}$ when $|T| \geq |\overline{T}|$ and $S = T \cup \{v_1, v_2\}$ when $|T| < |\overline{T}|$; so $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + 3 < m/3$, a contradiction. Thus $|\{v_4, v_5, v_6, v_7\} \cap \overline{T}| \leq 2$. Let $S = \overline{T} \cup \{u\}$ when $|\overline{T}| \geq |T|$ and $S = \overline{T} \cup \{u, v_1\}$ when $|\overline{T}| < |T|$. Then again $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + 3 < m/3$, a contradiction.

Thus, without loss of generality, we may assume $v_1v_3, v_2v_4 \in E(G)$ and $v_3 \in T$. If $v_4 \in \overline{T}$ then by symmetry, assume $|N(u) \cap T| \leq 2$, and let $S = T \cup \{u\}$ when $|T| \geq |\overline{T}|$ or $S = T \cup \{u, v_2\}$ when $|T| < |\overline{T}|$; so $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + 3 < m/3$, a contradiction. Hence $v_4 \in T$. If $\{v_5, v_6, v_7\} \subseteq \overline{T}$ then let $S = T \cup \{u\}$ when $|T| > |\overline{T}|$, and $S = T \cup \{v_1, v_2\}$ when $|T| \leq |\overline{T}|$; so $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + 3 < m/3$, a contradiction. Hence $\{v_5, v_6, v_7\} \not\subseteq \overline{T}$. Let $S = \overline{T} \cup \{u\}$ when $|\overline{T}| \geq |T|$ or $S = \overline{T} \cup \{u, v_1\}$ when $|\overline{T}| < |T|$; and again $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + 3 < m/3$, a contradiction.

Case 2. $h = 4$.

Suppose there exists $5 \leq i \leq 7$ such that $\{v_1, v_2, v_3, v_4\} \subseteq N(v_i)$, and without loss of generality let $i = 5$ and $v_5 \in T$. If $|T| < |\overline{T}|$ let $S = T \cup \{v_1, v_2, v_3\}$; otherwise let $S = T \cup \{u, v_1\}$ when $\{v_6, v_7\} \subseteq \overline{T}$, and $S = T \cup \{v_1, v_2\}$ when $\{v_6, v_7\} \not\subseteq \overline{T}$. It is easy to check that $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + 3 \leq (m-11)/3 + 3 < m/3$, a contradiction.

Next, suppose there exist $5 \leq i \neq j \leq 7$ such that $|N(v_i) \cap \{v_1, v_2, v_3, v_4\}| = 3$ and $|N(v_j) \cap \{v_1, v_2, v_3, v_4\}| = 1$. Without loss of generality, let $v_1, v_2, v_3 \in N(v_5)$, $v_4 \in N(v_6)$, and $v_5 \in T$. Suppose $v_6 \in T$. If $|T| \geq |\overline{T}|$ let $S = T \cup \{v_1, v_2\}$, and otherwise let $S = T \cup \{v_1, v_2, v_3\}$; then $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + 3 \leq (m-11)/3 + 3 < m/3$, a contradiction. So $v_6 \in \overline{T}$. Suppose $v_7 \in T$. If $|T| \geq |\overline{T}|$ let $S = T \cup \{v_1, v_4\}$, and otherwise let $S = T \cup \{v_1, v_2, v_4\}$; then $[S, \overline{S}]$ is a bisection such that $\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + 3 \leq (m-11)/3 < m/3$, a contradiction. So $v_7 \in \overline{T}$. If $|T| \geq |\overline{T}|$ let $S = T \cup \{u, v_1\}$, and otherwise let $S = T \cup \{v_2, v_3, v_4\}$. Then again, $[S, \overline{S}]$ is a bisection such that $\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + 3 \leq (m-11)/3 + 3 < m/3$, a contradiction.

Thus we may assume (by relabeling if necessary) that $\{v_1, v_2\} \subseteq N(v_5)$, $v_3 \in N(v_6)$, and $v_4 \notin N(v_5)$. By symmetry let $v_5 \in T$. Suppose $v_6 \in T$. If $|T| \geq |\overline{T}|$ let $S = T \cup \{v_3, v_4\}$, and otherwise let $S = T \cup \{v_1, v_2, v_3\}$; then $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + 3 \leq (m-11)/3 + 3 < m/3$, a contradiction. So $v_6 \in \overline{T}$. Similarly, $N(v_4) \setminus \{u\} \subseteq \overline{T}$. If $|T| \geq |\overline{T}|$ let $S = T \cup \{u, v_4\}$, and otherwise let $S = T \cup \{v_1, v_3, v_4\}$. Again, $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + 3 \leq (m-11)/3 + 3 < m/3$, a contradiction. ∎

Lemma 3.10. $|N(u) \cap V_2| \leq 3$ *for every* $u \in V_8$.

Proof. Let $u \in V_8$ and $N(u) = \{v_1, \ldots, v_8\}$. By Lemma 3.7(1), $|N(u) \cap V_2| \leq 4$. So assume for a contradiction that $N(u) \cap V_2 = \{v_1, \ldots, v_4\}$. Let $H = G - \{u, v_1, \ldots, v_4\}$. Then, $e(H) = m - 12$, and H is not a triangle as $|V(H)| \geq 4$. Note that $N(V_2) \subseteq \{u, v_5, v_6, v_7, v_8\}$ by Lemma 3.2. So if $\delta(H) \leq 1$ then for some $5 \leq i \leq 8$ we must have $|N(v_i) \setminus \{u, v_1, v_2, v_3, v_4\}| \leq 1$ and hence $d(v_i) \geq 6$ by Lemmas 3.3 and 3.6; so $|N(v_i) \cap V_2| \geq d(v_i) - 2 \geq (4d(v_i) - 6)/5$, contradicting

Lemma 3.7(1). Hence $\delta(H) \geq 2$, and H has a bisection $[T, \overline{T}]$ such that $\max\{e(T), e(\overline{T})\} < e(H)/3$.

Without loss of generality, assume $|T| \geq |\overline{T}|$ and $v_1v_5 \in E(G)$. If $v_5 \in T$ and $|\{v_6, v_7, v_8\} \cap T| \geq 2$ then let $S = T \cup \{v_2, v_3\}$; and otherwise, let $S = T \cup \{u, v_1\}$. In either case, $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + 4 < (m-12)/3 + 4 = m/3$. ∎

We now draw conclusions about V_k, $6 \leq k \leq 9$.

Lemma 3.11. *The following statements hold:*

(1) $e(V_2, V_6) = 0$*;*

(2) *if* $|V_2| \geq 3$ *then* $V_6 \cup V_7 = \emptyset$*;*

(3) *if* $|V_2| \geq 5$ *then* $V_8 = \emptyset$*;*

(4) *if* $|V_2| \geq 10$ *and four vertices in* V_2 *have the same neighborhood in* G*, then* $V_9 = \emptyset$*.*

Proof. Suppose there exists $uv \in E(G)$ with $u \in V_6$ and $v \in V_2$. By Lemmas 3.5 and 3.7(1), $|N(u) \cap V_2| = 2$; so $|V_1(G - \{u, v\})| = 1$ by Lemmas 3.1 and 3.3. Thus by adding an edge to $G - \{u, v\}$ incident with $V_1(G - \{u, v\})$, we obtain a graph H with $\delta(H) \geq 2$ and $e(H) = m - 6$. Moreover, H is not a triangle as $|V(H)| \geq 5$. So H has a bisection $[T, \overline{T}]$ such that $\max\{e(T), e(\overline{T})\} < e(H)/3$. By symmetry assume $|N(u) \cap \overline{T}| \leq 2$. Let $S = T \cup \{v\}$. Then $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + 2 < (m-6)/3 + 2 = m/3$, a contradiction. Hence we have (1).

To prove (2), let $u \in V_k$ and $k \in \{6, 7\}$, and let $x, y, z \in V_2$ be distinct and $H = G - \{u, x, y, z\}$. Note that if $k = 6$ then $N(u) \cap V_2 = \emptyset$ by (1), and if $k = 7$ then we may assume $N(u) \cap V_2 = \{x\}$ by Lemma 3.9. Then H is not a triangle (as $|V(H)| \geq 6$), $e(H) = m - 12$ (by Lemma 3.1), and $\delta(H) \geq 2$ (by Lemmas 3.1, 3.3 and 3.6). So H has a bisection $[T, \overline{T}]$ such that $\max\{e(T), e(\overline{T})\} < (m-12)/3$, and by symmetry assume $|N(u) \cap \overline{T}| \leq 3$. If $|N(u) \cap \overline{T}| \leq 2$, or $|N(u) \cap \overline{T}| = 3$ and $N(z) \cap T \neq \emptyset$, then let $S = T \cup \{x, y\}$; if $|N(u) \cap \overline{T}| = 3$ and $N(y) \cap T \neq \emptyset$ let $S = T \cup \{x, z\}$; and if $|N(u) \cap \overline{T}| = 3$ and $N(y) \cup N(z) \subseteq \overline{T}$ let $S = T \cup \{u, z\}$. Then $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + 4 < (m-12)/3 + 4 = m/3$, a contradiction. So (2) holds.

We now prove (3). Suppose $u \in V_8$ and let $N(u) = \{v_1, \ldots, v_8\}$, and let $w_1, \ldots, w_5 \in V_2$ be distinct. Let $N(u) \cap V_2 = \{v_1, \ldots, v_h\}$. By Lemma 3.10, $h \leq 3$; so we may assume $w_1, w_2 \notin N(u)$.

Suppose $h = 0$. Let $H = G - \{u, w_1, \ldots, w_5\}$. Then H is not a triangle (as $|V(H)| \geq 8$), $e(H) = m - 18$, and $\delta(H) \geq 2$ by Lemmas 3.1, 3.3 and 3.6 and by (1) and (2). Thus H has a bisection $[T, \overline{T}]$ such that $\max\{e(T), e(\overline{T})\} < (m-18)/3$, and we may assume $l := |N(u) \cap \overline{T}| \leq 4$. If $l \leq 2$, or $l = 3$ and $\min\{|N(w_4) \cap \overline{T}|, |N(w_5) \cap \overline{T}|\} \leq 1$, then let $S = T \cup \{w_1, w_2, w_3\}$; now $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + 6 < (m-18)/3 + 6 = m/3$, a contradiction. If $l = 3$ and $N(w_i) \subseteq \overline{T}$ for $i \in \{4, 5\}$, let $S = T \cup \{u, w_4, w_5\}$; then $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + 6 < (m-18)/3 + 6 = m/3$, a contradiction. So $l = 4$ and, hence, we have symmetry between T and $\overline{T}$; thus we may assume $e(\{w_4, w_5\}, \overline{T})| \geq 2$. Let $S = T \cup \{u, w_4, w_5\}$; then $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + 6 < (m-18)/3 + 6 = m/3$, a contradiction.

So $1 \leq h \leq 3$. Let $H = G - \{u, v_1, \ldots, v_h, w_1, \ldots, w_{h-1}\}$ if $h \geq 2$, and let $H = G - \{u, v_1\}$ if $h = 1$. Then H is not a triangle (as $|V(H)| \geq 5$), $e(H) = m - (8 + 3h - 2)$ (by Lemma 3.1), and $\delta(H) \geq 2$ by Lemmas 3.1, 3.3 and 3.6 and by (1) and (2). So let $[T, \overline{T}]$ be a bisection of H such that $\max\{e(T), e(\overline{T})\} < e(H)/3$, and by symmetry assume $|N(u) \cap \overline{T}| \leq (8-h)/2$. If $h = 1$ then let $S = T \cup \{v_1\}$; now $\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + 3 < (m-9)/3 + 3 = m/3$, a contradiction. Now assume $h = 2$. If $|N(u) \cap \overline{T}| \leq 2$ or $|N(w_1) \cap \overline{T}| \leq 1$ then let $S = T \cup \{v_1, v_2\}$, and otherwise let $S = T \cup \{u, w_1\}$; then $\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + 4 < (m-12)/3 + 4 = m/3$, a contradiction. So $h = 3$. If $|N(u) \cap \overline{T}| \leq 1$ or $e(\{w_1, w_2\}, \overline{T}) \leq 3$ let $S = T \cup \{v_1, v_2, v_3\}$, and otherwise let $S = T \cup \{u, w_1, w_2\}$; now $\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + 5 < (m-15)/3 + 5 = m/3$, a contradiction. So (3) is proved.

To prove (4), suppose $|V_2| \geq 10$ and four members of V_2 have the same neighborhood. Let $u \in V_9$. By Lemma 3.7(1), $|N(u) \cap V_2| \leq 5$. Since $|V_2| \geq 10$, $|V_2 \setminus N(u)| \geq |V_2 \cap N(u)|$; so $|N(u) \cap V_2| \leq 2$ by Lemma 3.7(2) (with $p = 1$). Hence, the four vertices in V_2 with the same neighborhood are all in $V_2 \setminus N(u)$. So by Lemma 3.7(3) (with $p = 2$), we have $|N(u) \cap V_2| \leq 1$.

Suppose $|N(u) \cap V_2| = 0$. Let $x_1, x_2, x_3 \in V_2 \setminus N(u)$ be distinct and have the same neighborhood. Let $H = G - \{u, x_1, x_2, x_3\}$. Then H is not a triangle (as $|V(H)| \geq 9$), $e(H) = m - 15$ (by Lemma 3.1), and $\delta(H) \geq 2$ (by Lemmas 3.1, 3.3 and 3.6). So let $[T, \overline{T}]$ be a bisection of H such that $\max\{e(T), e(\overline{T})\} < (m-15)/3$, and by symmetry assume $|N(u) \cap \overline{T}| \leq 4$. If $N(x_1) \cap T \neq \emptyset$ or $|N(u) \cap \overline{T}| \leq 3$ let $S = T \cup \{x_2, x_3\}$; otherwise let $S = T \cup \{u, x_1\}$. Then $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + 5 < (m-15)/3 + 5 = m/3$, a contradiction.

Thus, let $N(u) \cap V_2 = \{x\}$, and let $x_1, x_2, x_3, x_4 \in V_2 \setminus N(u)$ be distinct and have the same neighborhood in G. Let $H = G - \{u, x, x_1, x_2, x_3, x_4\}$. By Lemma 3.3, $N(V_2) \subseteq (N(u) \setminus \{x\}) \cup \{u\}$. Then $\delta(H) \geq 2$ (by Lemmas 3.1, 3.3 and 3.6 and by (2)), H is not a triangle (as $|V(H)| \geq 8$), and $e(H) = m - 18$ (by Lemma 3.1). So let $[T, \overline{T}]$ be a bisection of H such that $\max\{e(T), e(\overline{T})\} < (m-18)/3$, and by symmetry assume $|N(u) \cap \overline{T}| \leq 4$. Let $S = T \cup \{x, x_1, x_2\}$ if $N(x_3) \cap T \neq \emptyset$ or $|N(u) \cap \overline{T}| \leq 2$, and let $S = T \cup \{u, x_1, x_2\}$ otherwise. Then $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + 6 < (m-18)/3 + 6 = m/3$, a contradiction. ∎

We can now determine the parity of $|V(G)|$ which, in view of Corollary 2.4, will allow us to have a better control on $\Delta(G)$ and $e(G)$.

Lemma 3.12. *$|V(G)|$ is even.*

Proof. Suppose $n = |V(G)|$ is odd, and let M be a maximum symmetric matching in G ($M = \emptyset$ if there is none). Then by Corollary 2.4, $\delta(G) \leq 4$, $\Delta(G) \geq m/3 + |M|$, and $m \leq 3(n - |M| - 1)$.

We claim that $|V_3| = 0$. For, suppose there exists $u \in V_3$, and let $H = G - u$. Then $\delta(H) \geq 2$ (by Lemma 3.3), $e(H) = m - 3$, and H is not a triangle (by Lemma 2.1). Thus, H has a bisection $[T, \overline{T}]$ such that $\max\{e(T), e(\overline{T})\} < e(H)/3$, and by symmetry assume $|N(u) \cap T| \leq 1$. Let $S = T \cup \{u\}$. Note that $|T| = |\overline{T}|$ (as n is odd); so $[S, \overline{S}]$ is a bisection of G. Moreover, $\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + 1 < e(H)/3 + 1 \leq m/3$, a contradiction.

Next, we show that $\delta(G) = 4$. For, otherwise, $\delta(G) = 2$ as $|V_3| = 0$; hence $V_4 \cup V_5 = \emptyset$ by Lemma 3.6. First, assume $|V_2| \leq 2$. Then $2m \geq (m/3 + |M|) + 6(n-3) + 4$ and hence $m \geq 3.6n - 8.4 + 0.6|M|$; so $3(n-|M|-1) \geq 3.6n - 8.4 + 0.6|M|$, implying $n \leq 9 - 6|M|$. So by Lemma 2.1, $|M| = 0$ and $n \leq 9$. Now $n \neq 7$; otherwise, since $V_3 \cup V_4 \cup V_5 = \emptyset$, G has at least 5 vertices of degree 6, and hence $|M| > 0$, a contradiction. Thus, $n = 9$ as n is odd. Then $|V_2| = 2$ and $|V_6| \geq 6$, contradicting Lemma 3.11(1). Thus, there exist distinct $x, y, z \in V_2$, and let $H = G - \{x, y, z\}$. Then $e(H) = m - 6$ and $\delta(H) \geq 2$ by Lemma 3.1 and the fact $V_3 \cup V_4 \cup V_5 = \emptyset$. Since n is odd, $|V(H)|$ is even; so H is not a triangle. Hence H has a bisection $[T, \overline{T}]$ such that $|T| = |\overline{T}|$ and $\max\{e(T), e(\overline{T})\} < e(H)/3$. Without loss of generality, we may assume that $|N(x) \cap \overline{T}| \geq |N(x) \cap T|$ and $|N(y) \cap \overline{T}| \geq |N(y) \cap T|$. Let $S = T \cup \{x, y\}$. Then $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + 2 < e(H)/3 + 2 = m/3$, a contradiction.

We show now that $|V_4| \geq 3$. For, suppose $|V_4| \leq 2$. First, assume $V_5 = \emptyset$. Then $2m \geq (m/3 + |M|) + 6(n-3) + 8$ and so $m \geq 3.6n - 6 + 0.6|M|$; hence $3(n-|M|-1) \geq 3.6n - 6 + 0.6|M|$ which implies $n \leq 5$, contradicting Lemma 2.1. So $V_5 \neq \emptyset$. If $e(V_5) \neq 0$ then $|M| \geq 1$; so $2m \geq (m/3 + 1) + 5(n-3) + 8$ implying $m \geq 3n - 3.6$ which contradicts $m \leq 3(n - |M| - 1) \leq 3(n-2)$. Thus $e(V_5) = 0$. Then, since $|V_4| \leq 2$, $|N(v) \setminus (V_4 \cup V_5)| \geq 3$ for any $v \in V_5$. Hence at least three vertices of G have degree at least 6, and so $2m \geq (m/3 + |M|) + 5(n-5) + 20$. This implies $m \geq 3(n-1) + 0.6|M|$. Since $m \leq 3(n - |M| - 1)$, we must have $|M| = 0$ and $m = 3(n-1)$. Thus, all vertices in V_5 have the same five neighbors, two of degree 4, two of degree 6, and another of degree at least 6. It is then straightforward to check that $|M| > 0$, a contradiction.

Therefore, let $u_1, u_2, u_3 \in V_4$ be pairwise distinct, and let $H = G - \{u_1, u_2, u_3\}$. Then H is not a triangle as n is odd.

Suppose $e(V_4) = 0$. Then $\delta(H) \geq 2$ and $e(H) = m - 12$. So H has a bisection $[T, \overline{T}]$ such that $\max\{e(T), e(\overline{T})\} < e(H)/3$. Without loss of generality, assume $e(\{u_1, u_2\}, T) \leq 4$. Let $S = T \cup \{u_1, u_2\}$. Then $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + 4 < e(H)/3 + 4 = m/3$, a contradiction.

So $e(V_4) \geq 1$, and we may assume $u_1u_2 \in E(G)$. Now $e(H) \leq m - 9$. Suppose $\delta(H) \geq 2$. Then let $[T, \overline{T}]$ be a bisection of H such that $\max\{e(T), e(\overline{T}) < e(H)/3$, and without loss of generality assume $e(\{u_2, u_3\}, \overline{T}) \leq 3 - e(u_2, u_3)$. Let $S = T \cup \{u_1\}$. Then $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + 3 < e(H)/3 + 3 \leq m/3$, a contradiction.

Thus $\delta(H) = 1$, and let $u_4 \in V_1(H)$. Then $u_4 \in V_4(G)$ and $u_4u_i \in E(G)$ for $i = 1, 2, 3$. Let $K = G - \{u_1, u_2, u_4\}$. Then $e(K) = m - 9$ and K is not a triangle (as n is odd). Suppose $\delta(K) \geq 2$. Let $[T, \overline{T}]$ be a bisection of K such that $\max\{e(T), e(\overline{T})\} < e(K)/3$, and without loss of generality assume $e(\{u_2, u_4\}, \overline{T}) \leq 2$. Let $S := T \cup \{u_1\}$; then $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + 3 < e(H)/3 + 3 = m/3$, a contradiction.

So $\delta(K) = 1$, and let $u_5 \in V_1(K)$. Hence, $u_5 \in V_4(G)$ and $u_5u_1, u_5u_2, u_5u_4 \in E(G)$. Let $L = G - \{u_1, u_2, u_4, u_5\}$. Since $\delta(G) = 4$, $|V_1(L)| \leq 1$. So let $L' = L$ if $|V_1(L)| = 0$, and otherwise let L' be obtained from L by adding an edge from $V_1(L)$ to a vertex not adjacent to $V_1(L)$. Then $\delta(L') \geq 2$ and $e(L') \leq m - 9$. Note that L' is not a triangle (since $\delta(G) \geq 4$ and $u_1, u_2, u_4, u_5 \in V_4$). Thus, let $[T, \overline{T}]$ be a bisection of L' such that $\max\{e(T), e(\overline{T})\} < e(L')/3$. Let $S = T \cup \{u_1, u_2\}$. Now $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + 3 < e(L')/3 + 3 = m/3$, a contradiction. ∎

4 Vertices of degree 3

Let G be a minimum counterexample to Theorem 1.1, and for convenience let $e(G) = m$ and $|V(G)| = n$. We study properties of V_3, and show (in the order listed) that

- $e(V_3, V_5) = 0$,
- if $V_3 \neq \emptyset$ then $V_6 = \emptyset$,
- if $\delta(G) = 3$ then $V_7 = \emptyset$, and
- $|V_3| \leq 3$.

First, we prove two lemmas which give information about the neighbors of vertices in V_3.

Lemma 4.1. *If $uv \in E(G)$ and $u, v \in V_3$ then $G[(N(u) \cup N(v)) \setminus \{u, v\}]$ is complete.*

Proof. Let $uv \in E(G)$ with $u, v \in V_3$, $N(u) = \{u_1, u_2, v\}$, and $N(v) = \{v_1, v_2, u\}$. Suppose that $G[\{u_1, u_2\} \cup \{v_1, v_2\}]$ is not complete. Let $H = G - \{u, v\} + e$, where e is an edge joining two vertices in $\{u_1, v_1, u_2, v_2\}$ that are not adjacent in G and, moreover, e is incident with a vertex in $N(u) \cap N(v)$ whenever possible. Then $e(H) = m - 4$, and $\delta(H) \geq 2$ by Lemma 3.3. So H has a good bisection $[T, \overline{T}]$.

If $N(u) \cup N(v) \subseteq T$ then let $S = T \cup \{u\}$. Now $[S, \overline{S}]$ is a bisection of G, $e(\overline{S}) = e(\overline{T})) \leq e(H)/3 < m/3$ and $e(S) = e(T) - 1 + 2 \leq e(H)/3 + 1 < m/3$ (since $e \notin E(G)$), a contradiction. Hence, $N(u) \cup N(v) \not\subseteq T$. Similarly, $N(u) \cup N(v) \not\subseteq \overline{T}$.

Thus we may assume by symmetry that $|N(u) \cap T| \leq 1$ and $|N(v) \cap \overline{T}| \leq 1$. Let $S = T \cup \{u\}$. Then $[S, \overline{S}]$ is a bisection of G, and $\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + 1 \leq e(H)/3 + 1 < m/3$, a contradiction. ∎

Lemma 4.2. *If $u, v \in V_3$ and $uv \notin E(G)$ then $N(u) \cap N(v) \cap V_3 \neq \emptyset$.*

Proof. Let $u, v \in V_3$ such that $uv \notin E(G)$, and let $H = G - \{u, v\}$. Then $e(H) = m - 6$. Suppose $N(u) \cap N(v) \cap V_3 = \emptyset$. Then $\delta(H) \geq 2$ (by Lemma 3.3). By Lemma 3.12, H is not a triangle. So H has a bisection $[T, \overline{T}]$ such that $\max\{e(T), e(\overline{T})\} < e(H)/3$.

If $|N(u) \cap T| \leq 2$ and $|N(v) \cap \overline{T}| \leq 2$ then let $S = T \cup \{u\}$; now $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + 2 < e(H)/3 + 2 = m/3$, a contradiction. Similarly, we get a contradiction if $|N(v) \cap T| \leq 2$ and $|N(u) \cap \overline{T}| \leq 2$. Thus, by symmetry, assume $N(u) \subseteq \overline{T}$ and $N(v) \subseteq \overline{T}$.

Then $e(\overline{T}) \geq m/3 - 3$. For, otherwise, with $S = T \cup \{u\}$, $[S, \overline{S}]$ is a bisection of G such that $e(S) = e(T) < e(H)/3 < m/3$ and $e(\overline{S}) = e(\overline{T}) + 3 < (m/3 - 3) + 3 = m/3$, a contradiction.

If $|N(x) \cap \overline{T}| \leq 2$ for some $x \in T$ then, with $S = (T \setminus \{x\}) \cup \{u, v\}$, $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + 2 < e(H)/3 + 2 = m/3$, a contradiction. So $|N(x) \cap \overline{T}| \geq 3$ for all $x \in T$. Thus, $m \geq 3(n-2)/2 + (m/3 - 3) + 6$; hence $m \geq 9n/4$.

By Lemma 3.12 and Corollary 2.4(2), $\Delta(G) \geq m/3 + 2$ and $m \leq 3(n - 3)$. Since

$$m = e(T) + e(\overline{T}) + e(T, \overline{T}) + 6 \geq e(T) + (m/3 - 3) + 3(n - 2)/2 + 6,$$

$e(T) \le 2m/3 - 3n/2$. If T contains a vertex of degree $\Delta(G)$ then

$$m = e(T) + e(T, \overline{T}) + e(\overline{T}) + 6 \ge 3(n-4)/2 + (m/3 + 2) + (m/3 - 3) + 6,$$

and so $m \ge 9n/2 - 3$, a contradiction as $m \le 3(n-3)$. Thus, $\overline{T}$ contains a vertex, say w, of degree $\Delta(G)$. Let $S = T \cup \{w\}$. Then $[S, \overline{S}]$ is a bisection of G. Since $m \le 3(n-3)$, $n \ge m/3 + 3$; so

$$e(S) = e(T) + e(w, T) \le (2m/3 - 3n/2) + (n-2)/2 = 2m/3 - n - 1 < m/3.$$

Thus,

$$m/3 \le e(\overline{S}) \le e(\overline{T}) - e(w, \overline{S}) + 6 < (m-6)/3 - (m/3 + 2 - (n-2)/2) + 6 = n/2 + 1.$$

Hence $n/2 + 1 > m/3 \ge 3n/4$ (since $m \ge 9n/4$); so $n < 4$, contradicting Lemma 2.1. ∎

We can now prove the following

Lemma 4.3. *$e(V_3, V_5 \cup V_6) = \emptyset$ and, moreover, $V_3 = \emptyset$ or $V_6 = \emptyset$.*

Proof. Suppose there exists $uv \in E(G)$ such that $u \in V_3$ and $v \in V_5 \cup V_6$. By Lemmas 3.3, 3.6 and 3.11(1), $N(u) \cap V_2 = N(v) \cap V_2 = \emptyset$. So $\delta(G - \{u, v\}) \ge 1$ and $V_1(G - \{u, v\}) \subseteq N(u) \cap N(v) \cap V_3$. Let $N(u) = \{v, u_1, u_2\}$.

If $V_1(G - \{u, v\}) = \{u_1, u_2\}$ then let $H = G - \{u, v, u_1, u_2\}$. Now $\delta(H) \ge 2$ and $e(H) \le m - 7$. So H has a good bisection $[T, \overline{T}]$, and by symmetry assume $|N(v) \cap T| \le 1$. Let $S = T \cup \{u, v\}$. Then $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \le \max\{e(T), e(\overline{T})\} + 2 \le e(H)/3 + 2 < m/3$, a contradiction.

So $\{u_1, u_2\} \not\subseteq V_1(G - \{u, v\})$. Thus by adding at most one edge to $G - \{u, v\}$, we obtain a graph H with $\delta(H) \ge 2$ and $e(H) \le m - 6$. Since n is even, H is not a triangle. So H has a bisection $[T, \overline{T}]$ such that $\max\{e(T), e(\overline{T})\} < e(H)/3$, and by symmetry assume $|N(v) \cap \overline{T}| \le 2$. Let $S = T \cup \{u\}$. Then $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \le \max\{e(T), e(\overline{T})\} + 2 < e(H)/3 + 2 \le m/3$, a contradiction.

Thus, we have shown $e(V_3, V_5 \cup V_6) = 0$. Now, suppose there exist $u \in V_3$ and $v \in V_6$, and let $H = G - \{u, v\}$. Then, $e(H) = m - 9$ (since $e(V_3, V_6) = 0$), and $\delta(H) \ge 2$ (by Lemmas 3.3 and 3.11(1)). Since n is even, H is not a triangle. So H has a bisection $[T, \overline{T}]$ such that $\max\{e(T), e(\overline{T})\} < e(H)/3$, and by symmetry assume $|N(v) \cap \overline{T}| \le 3$. Let $S = T \cup \{u\}$. Then, $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \le \max\{e(T), e(\overline{T})\} + 3 < e(H)/3 + 3 = m/3$, a contradiction. ∎

We need a lemma to study the relation between V_3 and V_k for $k \ge 4$. A *wheel of order* $k \ge 4$ is the graph obtained from a cycle of length $k-1$ by adding a vertex (called the *center*) and adding an edge between the center and each vertex on the cycle. Thus, if $uv \in E(G)$ with $u, v \in V_3$ then by Lemma 4.1, $G[N(u) \cup N(v)]$ is a wheel of order 4 when $|N(u) \cap N(v)| = 2$, a wheel of order 5 when $|N(u) \cap N(v)| = 1$, and $G[(N(u) \cup N(v)) \setminus \{u, v\}]$ is a K_4 when $N(u) \cap N(v) = \emptyset$.

Lemma 4.4. *Suppose W is a wheel with center u in G such that $|V(W)| \in \{4, 5\}$ and $|V(W - u) \cap V_3| \ge 3$. Then $\delta(G) = 3$, $d(u) \ge 10$ and $|V_3| = 3$.*

Proof. First, suppose $|V(W-u) \cap V_3| \neq 3$. Then $|V(W)| = 5$ and $V(W-u) \subseteq V_3$. Let $v_1v_2v_3v_4v_1$ be the cycle $W-u$. Let $H = G-\{v_1, v_2, v_3, v_4\}$. Then $e(H) = m-8$. By Lemmas 2.1 and 4.3, $d(u) \geq 7$; so $\delta(H) \geq 2$. Hence H has a good bisection, say $[T, \overline{T}]$. Let $S = T \cup \{v_1, v_3\}$. Now $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \leq (m-8)/3+2 < m/3$, a contradiction.

So $|V(W-u) \cap V_3| = 3$. Let $v_1v_2v_3v_4v_1$ (when $|V(W)| = 5$) or $v_1v_2v_3v_1$ (when $|V(W)| = 4$) denote the cycle $W-u$ such that $v_1, v_2, v_3 \in V_3$.

Now suppose $V_2 \neq \emptyset$ and let $x \in V_2$. Then $x \notin W$, $N(x) \cap \{v_1, v_2, v_3\} = \emptyset$, and $d(u) \geq 7$ (by Lemmas 3.6 and 4.3). If $|V(W)| = 5$, let $H = G - \{v_1, v_2, v_3, x\}$; then $e(H) = m-9$, $\delta(H) \geq 2$ (by Lemma 3.1), and H is not a triangle (as n is even). Thus H has a bisection $[T, \overline{T}]$ such that $\max\{e(T), e(\overline{T})\} < e(H)/3$. By symmetry assume $u \in T$. Then, with $S = T \cup \{v_2, x\}$, $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + 3 < e(H)/3 + 3 \leq m/3$, a contradiction. Therefore, $|V(W)| = 4$. Let $y \in N(x) \setminus \{u\}$, and let $H = G - \{v_1, v_2\} + \{v_3x, v_3y\}$. Then $e(H) = m-3$, $\delta(H) \geq 2$ and H is not a triangle (as n is even). So H has a bisection $[T, \overline{T}]$ such that $\max\{e(T), e(\overline{T})\} < e(H)/3$, and by symmetry assume $u \in T$. If $v_3 \in \overline{T}$ then let $S = T \cup \{v_1\}$; now $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + 1 < e(H)/3 + 1 = m/3$, a contradiction. So $v_3 \in T$. Let $S = T \cup \{v_1\}$ if $\{x, y\} \not\subseteq \overline{T}$, and let $S = T \cup \{x\}$ otherwise. Then, $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + 1 < e(H)/3 + 1 = m/3$, a contradiction.

Thus $V_2 = \emptyset$ and $\delta(G) = 3$. By Lemma 4.3, $d(u) \notin \{5, 6\}$. If $\delta(G - \{u, v_1, v_2, v_3\}) \geq 2$ then let $H = G - \{u, v_1, v_2, v_3\}$; otherwise $|V(W)| = 5$ and $V_1(G - \{u, v_1, v_2, v_3\}) = \{v_4\}$, and let H denote the graph obtained from $G - \{u, v_1, v_2, v_3\}$ by adding an edge from v_4 to a vertex not adjacent to v_4. Then $e(H) \leq m - d(u) - 3$, $\delta(H) \geq 2$, and H is not a triangle (since n is even). Let $[T, \overline{T}]$ be a bisection of H such that $\max\{e(T), e(\overline{T})\} < e(H)/3$, and by symmetry assume $|N(u) \cap \overline{T}| \leq \lfloor (d(u)-3)/2 \rfloor$. Let $S = T \cup \{v_1, v_3\}$. Then $e(S) < e(H)/3 + 2 \leq m/3$, and so

$$m/3 \leq e(\overline{S}) < (m - d(u) - 3)/3 + (d(u) - 3)/2 + 1 = m/3 + (d(u) - 9)/6.$$

Hence $d(u) \geq 10$.

Finally, suppose $|V_3| \geq 4$ and let $x \in V_3 \setminus \{v_1, v_2, v_3\}$. Then, $v_2x \notin E(G)$ and $N(v_2) \cap N(x) \cap V_3 = \emptyset$, contradicting Lemma 4.2. ∎

We also need a lemma about edges from V_3 to V_4.

Lemma 4.5. *Let $uv \in E(G)$ with $u \in V_3$ and $v \in V_4$. Then $\delta(G) = 3$, $|V_3| = 2$, and $N(u) \cap N(v)$ consists of two adjacent vertices, one in V_3 and one with degree at least 12.*

Proof. By Lemma 3.6, $V_2 = \emptyset$; so $\delta(G) = 3$. Suppose $N(u) \cap N(v) \cap V_3 = \emptyset$, and let $H = G - \{u, v\}$. Then $\delta(H) \geq 2$, $e(H) = m - 6$, and H is not a triangle (as n is even). So H has a bisection $[T, \overline{T}]$ such that $\max\{e(T), e(\overline{T})\} < e(H)/3$, and by symmetry assume $|N(v) \cap \overline{T}| \leq 1$. Let $S = T \cup \{u\}$. Then $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + 2 < e(H)/3 + 2 = m/3$, a contradiction.

So $N(u) \cap N(v) \cap V_3 \neq \emptyset$, and let $x \in N(u) \cap N(v) \cap V_3$. Let $N(u) = \{v, x, u_1\}$ and $N(x) = \{u, v, x_1\}$. By Lemma 4.1, $vx_1, vu_1 \in E(G)$, and $u_1x_1 \in E(G)$ if $u_1 \neq x_1$. Hence, $N(u) \cap N(v) = \{u_1, x\}$.

Suppose $x_1 \neq u_1$, and let $H = G - \{u, v\} + u_1 x$. Then $\delta(H) \geq 2$ and $e(H) = m-5$. So H has a good bisection $[T, \overline{T}]$, and by symmetry assume $x \in T$. Let $S = T \cup \{v\}$ when $\{u_1, x_1\} \subseteq \overline{T}$, and let $S = T \cup \{u\}$ otherwise. Then $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + 1 \leq e(H)/3 + 1 < m/3$ (since $u_1 x \notin E(G)$), a contradiction.

Thus $x_1 = u_1$. So $u_1 x \in E(G)$. By Lemmas 4.3 and 4.4, $u_1 \notin V_3 \cup V_5 \cup V_6$. Then by Lemma 4.2, $V_3 = \{u, x\}$, and hence $|V_3| = 2$.

We now prove $d(u_1) \geq 12$. Suppose, for contradiction, $d(u_1) \leq 11$. Let $H = G - \{u, u_1, v, x\}$. Since $V_2 = \emptyset$ and $|V_3| = 2$, $\delta(H) \geq 2$. Moreover, H is not a triangle as n is even. So H has a bisection $[T, \overline{T}]$ such that $\max\{e(T), e(\overline{T})\} < e(H)/3 = (m - d(u_1) - 4)/3$. By symmetry, assume $|N(u_1) \cap \overline{T}| \leq (d(u_1) - 3)/2$. Let $S = T \cup \{u, v\}$. Then, $[S, \overline{S}]$ is a good bisection of G, $e(S) \leq e(T) + 2 < (m - d(u_1) - 4)/3 + 2 < m/3$, and

$$e(\overline{S}) \leq e(\overline{T}) + (d(u_1) - 3)/2 + 1 < (m - d(u_1) - 4)/3 + (d(u_1) - 1)/2 = m/3 + (d(u_1) - 11)/6.$$

Since $d(u_1) \leq 11$, $e(\overline{S}) < m/3$, a contradiction. ∎

We can now prove the following

Lemma 4.6. *If $\delta(G) = 3$ then $V_7 = \emptyset$.*

Proof. For, suppose $\delta(G) = 3$ and $V_7 \neq \emptyset$. Let $u \in V_3$ and $v \in V_7$ with $uv \in E(G)$ when possible. Then $|N(u) \cap N(v) \cap V_3| \geq 1$. For, otherwise, $H = G - \{u, v\}$ has $e(H) \leq m - 9$ and $\delta(H) \geq 2$. Since n is even, H is not a triangle. So H has a bisection $[T, \overline{T}]$ such that $\max\{e(T), e(\overline{T})\} < e(H)/3$. By symmetry, assume $|N(v) \cap \overline{T}| \leq 3$. Let $S = T \cup \{u\}$; then $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + 3 < e(H)/3 + 3 \leq m/3$, a contradiction

Hence $uv \in E(G)$ by the choice of u and v. Let $w \in N(u) \cap N(v) \cap V_3$, and let $H = G - \{u, v, w\}$. Then $e(H) = m - 10$. By Lemma 4.1, $G[N(u) \cup N(w)]$ is a wheel of order 4 when $N(u) \cap N(w) \neq \{v\}$, or a wheel of order 5 when $N(u) \cap N(w) = \{v\}$. Thus, $\delta(H) \geq 2$ by Lemmas 4.4 and 4.5 (since $v \in V_7$). So H has a good bisection $[T, \overline{T}]$, and by symmetry assume $|T| < |\overline{T}|$ (as n is even by Lemma 3.12). Let $S = T \cup \{u, w\}$ when $|N(v) \cap \overline{T}| \leq 3$, and let $S = T \cup \{u, v\}$ otherwise. Then $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \leq \max\{e(T), e(\overline{T})\} + 3 \leq (m - 10)/3 + 3 < m/3$, a contradiction. ∎

We conclude this section by proving $|V_3| \leq 3$.

Lemma 4.7. *$|V_3| \leq 3$, if $|V_3| = 2$ then $e(V_3) = 1$, and if $|V_3| = 3$ then G contains a wheel W with center u such that $4 \leq |V(W)| \leq 5$ and $V_3 \subseteq V(W - u)$.*

Proof. If $|V_3| = 2$ then $e(V_3) = 1$ by Lemma 4.2. So assume $|V_3| \geq 3$.

If $G[V_3]$ is complete then, by Lemma 2.1, $|V_3| = 3$, and it follows from Lemma 4.1 that G contains a wheel W with center u such that $|V(W)| = 4$ and $V_3 = V(W - u)$. So assume that there exist $x, y \in V_3$ such that $xy \notin E(G)$.

By Lemma 4.2, there exists $z \in N(x) \cap N(y) \cap V_3$. Write $N(x) = \{u, v, z\}$, $N(z) = \{w, x, y\}$, and $A = \{u, v, w, y\}$. Then by Lemma 4.1, $G[A]$ is complete. Since $y \in V_3$, we must have $N(y) = \{u, v, z\}$ and $w \in \{u, v\}$. Without loss of generality, assume $w = u$. Hence, $W = G[\{u, v, x, y, z\}]$ is a wheel with center u, and $|V_3 \cap V(W - u)| \geq 3$. So by Lemma 4.4, $V_3 = V(W - u) \cap V_3 = \{x, y, z\}$. ∎

5 Lower bounds

Let G be a counterexample to Theorem 1.1 with $|V(G)|$ minimum, and for convenience let $m = e(G)$ and $n = |V(G)|$. In this section, we will prove lower bounds on m, n, $|V_2 \cup V_3|$, and $|V_2|$. By Lemma 3.12 and Corollary 2.4, $\Delta(G) \geq m/3 + |M| + \delta(G) \geq m/3 + |M| + 2$ and $m \leq 3(n - |M| - \delta(G) - 1) \leq 3(n - |M| - 3)$, where M is any symmetric matching in G. By Lemmas 2.1 and 3.12, $n \geq 8$. First, we give a lower bound on $|V_2 \cup V_3|$.

Lemma 5.1. *$|V_2 \cup V_3| \geq 2$.*

Proof. Let $V(G) = \{v_1, \ldots, v_n\}$ such that $d(v_1) \leq \ldots \leq d(v_n)$. If $d(v_2) \leq 3$, we are done. So assume $d(v_2) \geq 4$. If $d(v_2) \geq 5$ then $2m \geq (m/3 + 2) + 5(n - 2) + 2$; so $m \geq 3(n - 1.2)$, a contradiction as $m \leq 3(n - 3)$. So $d(v_2) = 4$. Thus, $d(v_1) \in \{3, 4\}$ by Lemma 3.6. Let $H = G - \{v_1, v_2\}$. Then $\delta(H) \geq 2$, $e(H) \leq m - 6$, and H is not a triangle (since n is even). So H has a bisection $[T, \overline{T}]$ such that $\max\{e(T), e(\overline{T})\} < e(H)/3$.

If there exists $i \in \{1, 2\}$ such that $|N(v_i) \cap T| \leq 2$ and $|N(v_{3-i}) \cap \overline{T}| \leq 2$, then with $S = T \cup \{v_i\}$, $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S}\} \leq \max\{e(T), e(\overline{T})\} + 2 < (m-6)/3+2 = m/3$, a contradiction. Thus by symmetry, we may assume that $|N(v_i) \cap \overline{T}| \geq 3$ for $i \in \{1, 2\}$. Hence, $v_1 v_2 \notin E(G)$ if $v_1 \in V_3$.

Since $\delta(G) = d(v_1) \geq 3$, $\Delta(G) \geq m/3 + 3$ and $m \leq 3(n - 4)$. Moreover, since $d(v_2) = 4$, $2m \geq (m/3 + 3) + 4(n - 2) + 3$; so $m/3 \geq (4n - 2)/5$. Thus $n - 4 \geq (4n - 2)/5$; so $n \geq 18$. Hence $m/3 \geq (4n - 2)/5 > (n + 4)/2$.

Note that $e(\overline{T}) \geq m/3 - 4$; otherwise, with $S = T \cup \{v_1\}$, $[S, \overline{S}]$ is a bisection of G such that $e(S) \leq e(T)+1 < (m-6)/3+1 < m/3$ and $e(\overline{S}) \leq e(\overline{T})+4 < m/3$, a contradiction. Also $|N(x) \cap \overline{T}| \geq 3$ for every $x \in T$; otherwise, with $S = (T \setminus \{x\}) \cup \{v_1, v_2\}$, $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S}\} \leq \max\{e(T), e(\overline{T})\} + 2 < (m - 6)/3 + 2 = m/3$, a contradiction. Hence $e(T, \overline{T}) \geq 3(n - 2)/2$. Since

$$m \geq e(T) + e(T, \overline{T}) + e(\overline{T}) + 7 \geq e(T) + 3(n - 2)/2 + (m/3 - 4) + 7,$$

$e(T) \leq 2m/3 - 3n/2$. Moreover, there exists $x_0 \in T$ such that $|N(x_0) \cap \overline{T}| = 3$; otherwise, $e(T, \overline{T}) \geq 2(n - 2)$ and $m \geq (m/3 - 4) + 2(n - 2) + 7$, yielding $m > 3n - 3/2$, a contradiction (as $m \leq 3(n - 4)$). Further, $v_n \notin T$; for otherwise,

$$\begin{aligned} m &\geq e(T \setminus \{v_n\}, \overline{T}) + e(\overline{T}) + d(v_n) + d(v_1) + d(v_2) - e(\{v_1, v_2\}, \{v_n\}) - e(v_1, v_2) \\ &\geq 3(n - 4)/2 + (m/3 - 4) + (m/3 + 3) + 4; \end{aligned}$$

so $m \geq 4.5n - 9$ and hence $3(n - 4) \geq 4.5n - 9$, a contradiction. Let $S = (T \setminus \{x_0\}) \cup \{v_1, v_n\}$. Then $[S, \overline{S}]$ is a bisection of G,

$$e(S) \leq e(T) + (n - 4)/2 + 2 \leq (2m/3 - 3n/2) + (n - 4)/2 + 2 = 2m/3 - n,$$

and

$$e(\overline{S}) \le e(\overline{T}) - (m/3 + 3 - (n-2)/2 - 1) + 7 < (m-6)/3 - (m/3 + 3 - n/2) + 7 = (n+4)/2.$$

Since $m \le 3(n-4)$, $n \ge m/3 + 4$; so $e(S) \le m/3 - 4 < m/3$. Further, since $m/3 > (n+4)/2$, $e(\overline{S}) < m/3$. Thus, $\max\{e(S), e(\overline{S})\} < m/3$, a contradiction ∎

Next, we bound $|V_2|$.

Lemma 5.2. *$|V_2| = 0$ or $|V_2| \ge (4n-4)/7$.*

Proof. Suppose $|V_2| \ne 0$. Then $|V_3| \le 2$ by Lemmas 4.7 and 4.4, and $V_4 = V_5 = \emptyset$ by Lemma 3.6. First, we show $|V_2| \ge 3$. If $V_3 = \emptyset$ then

$$2m \ge (m/3 + 2) + 6(n - |V_2| - 1) + 2|V_2|,$$

and hence $4|V_2| \ge 6n - 4 - 5m/3$; so $|V_2| \ge (n+11)/4 \ge 4$ (since $m \le 3(n-3)$ and $n \ge 8$). Now assume $V_3 \ne \emptyset$. Then $V_6 = \emptyset$ by Lemma 4.3. Hence

$$2m \ge (m/3 + 2) + 7(n - |V_2| - 3) + 6 + 2|V_2|;$$

so $5|V_2| \ge 7n - 13 - 5m/3$. Since $m \le 3(n-3)$ and $n \ge 8$, $|V_2| \ge (2n+2)/5 \ge 3$.

So by Lemma 3.11(2), $V_6 \cup V_7 = \emptyset$. Thus,

$$2m \ge (m/3 + 2) + 8(n - |V_2| - 3) + 6 + 2|V_2|;$$

so $6|V_2| \ge 8n - 16 - 5m/3$ yielding $|V_2| \ge (3n-1)/6$ (since $m \le 3(n-3)$). Since $\Delta(G) \ge m/3 + 2 > 3$, $\Delta(G) \ge 8$; so $n \ge 10$ (as n is even). Hence, $|V_2| \ge 5$, and $V_8 = \emptyset$ by Lemma 3.11(3). Therefore,

$$2m \ge (m/3 + 2) + 9(n - |V_2| - 3) + 6 + 2|V_2|.$$

So $7|V_2| \ge 9n - 19 - 5m/3$; so $|V_2| \ge (4n-4)/7$ as $m \le 3(n-3)$. ∎

We now bound $m = e(G)$ from below. By Lemma 5.1, $\delta(G) = 2$ or 3.

Lemma 5.3. *If $\delta(G) = 3$ then $m \ge 3(4n-3)/5$.*

Proof. Suppose $\delta(G) = 3$. Then $\Delta(G) \ge m/3 + 3$ (by Corollary 2.4), and $|V_3| \le 3$ (by Lemma 4.7). If $|V_3| = 1$ then $2m \ge (m/3+3) + 4(n-2) + 3$; so $m \ge 3(4n-2)/5$. Now assume $|V_3| \ge 2$. Then $e(V_3) \ge 1$ by Lemma 4.7, and so G has a nonempty symmetric matching. Hence $\Delta(G) \ge m/3 + 4$ by Corollary 2.4. So $2m \ge (m/3+4) + 4(n-4) + 9$, giving $m \ge 3(4n-3)/5$. ∎

Lemma 5.4. *If $\delta(G) = 2$ then $n \ge 10$ and $m \ge 3(20n-21)/29$.*

Proof. Suppose $\delta(G) = 2$. Then $V_4 = V_5 = \emptyset$ by Lemma 3.6. By Lemma 5.2, $|V_2| \geq (4n - 4)/7 \geq 4$ (as $n \geq 8$). Thus $V_6 = V_7 = \emptyset$ by Lemma 3.11(2). By Corollary 2.4, $\Delta(G) \geq m/3+2 \geq 4$; so $\Delta(G) \geq 8$, which implies $n \geq 10$ (as n is even). Therefore, $|V_2| \geq (4n-4)/7 > 5$, and $V_8 = \emptyset$ by Lemma 3.11(3).

We now apply a simple discharging procedure to G. Let the degree $d(x)$ be the initial charge at each $x \in V(G)$. For each $uv \in E(G)$ with $u \in V(G) \setminus (V_2 \cup V_3)$ (so $d(u) \geq 9$) and $v \in V_2$, u sends 1 unit of charge to v. For each $x \in V(G)$, let $\omega(x)$ denote the final charge at x. Then

$$2m = \sum_{x \in V(G)} \omega(x).$$

We claim that $\omega(x) \geq 4$ for all $x \in V(G) \setminus V_3$. If $x \in V_2$ then by Lemmas 3.1 and 3.3, $N(x) \cap (V_2 \cup V_3) = \emptyset$; so $\omega(x) = 4$. Now assume $x \notin V_2$. Then by Lemma 3.7(1), $|N(x) \cap V_2| < (4d(x) - 6)/5$, and hence $\omega(x) > d(x) - (4d(x)-6)/5 = (d(x)+6)/5$. Since $d(x) \geq 9$, $\omega(x) \geq 4$.

By Lemmas 4.4 and 4.7, $|V_3| \leq 2$, and if $|V_3| = 2$ then $e(V_3) = 1$. If $|V_3| = 2$ then G has a nonempty symmetric matching, and hence $\Delta(G) \geq m/3 + 3$ by Corollary 2.4; so

$$2m = \sum_{x \in V(G)} \omega(x) \geq 4(n-3) + 6 + ((m/3+3)+6)/5,$$

yielding $m \geq 3(20n-21)/29$. So assume $|V_3| \leq 1$. Then

$$2m = \sum_{x \in V(G)} \omega(x) \geq 4(n-2) + 3 + ((m/3+2)+6)/5,$$

yielding $m > (60n-51)/29 > 3(20n-21)/29$. ∎

Finally, we establish a lower bound on $n = |V(G)|$.

Lemma 5.5. $|V(G)| \geq 12$.

Proof. If $\delta(G) = 3$ then by Corollary 2.4 and Lemma 5.3, $3(4n-3)/5 \leq 3(n-4)$; so $n \geq 17$. Now assume $\delta(G) = 2$. Then $n \geq 10$ by Lemma 5.4. So $|V_2| \geq 6$ by Lemma 5.2. If $n = 10$ then $m \geq 3(20n-21)/29 > 18$. However, $e(V_2) = 0$ by Lemma 3.1; so m is maximum when $G - V_2$ is complete. Thus, $m \leq 12 + 6 = 18$, a contradiction. Hence $n \geq 12$ as n is even. ∎

6 A bound on maximum bisection

To deal with the remaining case when a minimum counterexample G to Theorem 1.1 has number of edges between $3(20|V(G)| - 21)/29$ (or $3(4|V(G)|-3)/5$) and $3(|V(G)|-3)$, we need an idea from Lee, Loh and Sudakov [13].

Let H be a graph. Let $M = \{e_1, \ldots, e_s\}$ be a maximum matching in H, let $V(M)$ denote the set of vertices covered by M, and let $W = V(H) \setminus V(M)$. For $xy \in M$ and $z \in W$, x is said to be a *free neighbor* of z with respect to M if $xz \in E(H)$ but $yz \notin E(H)$. A vertex in W is an *M-free vertex* if it has at least one free neighbor with respect to M.

The proof of Theorem 1.3 in [13] implies the following result, which we will use in our proof of Theorem 1.1. For the sake of completeness, we give a proof here.

Lemma 6.1. *Let H be a graph, M a maximum matching in H, and $\{u_1, \ldots, u_r, v_1, \ldots, v_r\} \subseteq V(H) \setminus V(M)$ such that for each $1 \le i \le r$, u_i and v_i have different sets of free neighbors (with $r = 0$ when the subset is empty). Then H admits a bisection $[S, \overline{S}]$ such that $e(S, \overline{S}) \ge (e(H) + |M| + r)/2$.*

Proof. For convenience, assume $|V(H)|$ is even (by adding an isolated vertex to H if necessary). Let $M = \{x_1y_1, \ldots, x_sy_s\}$, and let $U = V(M) \cup \{u_1, \ldots, u_r, v_1, \ldots, v_r\}$. We will construct a bisection of H of size at least $(e(H) + s + r)/2$. Since M is a maximum matching in H, $V(H) \setminus V(M)$ is an independent set, and for any $a, b \in V(H) \setminus V(M)$ and any $x_iy_i \in M$, $\{x_ia, y_ib\} \not\subseteq E(G)$.

Arbitrarily partition the vertices in $V(H) \setminus U$ into pairs $\{a_1, b_1\}, \ldots, \{a_t, b_t\}$. Order the pairs $\{x_i, y_i\}$, $\{u_j, v_j\}$ and $\{a_k, b_k\}$ into a sequence as follows. Arrange $\{x_1, y_1\}, \ldots, \{x_s, y_s\}$ in this order. For each $1 \le j \le r$, since u_j and v_j have different sets of free neighbors with respect to M, there must be a smallest index i such that exactly one of u_j and v_j has a free neighbor in $\{x_i, y_i\}$; we insert $\{u_j, v_j\}$ between $\{x_i, y_i\}$ and $\{x_{i+1}, y_{i+1}\}$, and if several $\{u_j, v_j\}$ are inserted between $\{x_i, y_i\}$ and $\{x_{i+1}, y_{i+1}\}$ then these $\{u_j, v_j\}$ are arranged in their natural order. After arranging all $\{x_i, y_i\}$ and $\{u_j, v_j\}$, we append $\{a_1, b_1\}, \ldots, \{a_t, b_t\}$ to the end of the resulting sequence, and let $Q_1, Q_2, \ldots, Q_l$ denote the final sequence, where $l = r + s + t$.

We now form a bisection of H by partitioning Q_i in the order $i = 1, \ldots, l$. Let $S_0 = T_0 = \emptyset$. For each $i \in \{1, \ldots, k\}$, we construct S_i and T_i as follows.

(a) $S_{i-1} \subseteq S_i$, $T_{i-1} \subseteq T_i$, and $|S_i \cap Q_i| = |T_i \cap Q_i| = 1$

(b) subject to (a), $e(S_i, T_i)$ is maximum.

Let $S = S_l$; so $\overline{S} = T_l$. We show that $e(S, \overline{S}) \ge (e(H) + s + r)/2$.

Let $H_i = H[S_i \cup T_i]$, the subgraph of H induced by $S_i \cup T_i$. Then by (b), $e(S_i, T_i) - e(S_{i-1}, T_{i-1}) \ge (e(H_i) - e(H_{i-1}))/2$. For $1 \le i \le r + s$, either $Q_i = \{x_i, y_i\}$ or $Q_i = \{u_j, v_j\}$ for some j with $d_{H_i}(u_j) + d_{H_i}(v_j)$ odd; hence

$$e(S_i, T_i) - e(S_{i-1}, T_{i-1}) \ge (e(H_i) - e(H_{i-1}) + 1)/2.$$

Therefore,

$$\begin{aligned} e(S, \overline{S}) &= e(S_1, T_1) + \sum_{i=2}^{l} (e(S_i, T_i) - e(S_{i-1}, T_{i-1})) \\ &\ge e(H_1) + \frac{1}{2} \sum_{i=2}^{r+s} (e(H_i) - e(H_{i-1}) + 1) + \frac{1}{2} \sum_{i=r+s+1}^{l} (e(H_i) - e(H_{i-1})) \\ &= \frac{e(H) + s + r}{2}. \end{aligned}$$

Thus $[S, \overline{S}]$ is the desired bisection. ∎

7 Proof of Theorem 1.1

Let G be a minimum counterexample to Theorem 1.1, and let $m = e(G)$ and $n = |V(G)|$. By Lemmas 3.12 and 5.5, n is even and $n \geq 12$. By Corollary 2.4, for any symmetric matching M in G,

$$\Delta(G) \geq m/3 + |M| + \delta(G) \geq m/3 + 2$$

and

$$m \leq 3(n - |M| - \delta(G) - 1) \leq 3(n - 3).$$

Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ such that $d(v_1) \leq d(v_2) \leq \ldots \leq d(v_n)$. Moreover, we may assume that $v_1v_2 \in E(G)$ whenever G has an edge joining a vertex of degree $d(v_1)$ and a vertex of degree $d(v_2)$. By Lemma 5.1,

$$d(v_2) \leq 3.$$

Let d denote the number of edges of G incident with $\{v_1, v_2\}$. Then $d = d(v_1) + d(v_2)$ when $v_1v_2 \notin E(G)$, and $d = d(v_1) + d(v_2) - 1$ otherwise. So by Lemma 4.2 and the choices of v_1 and v_2, $v_1v_2 \in E(G)$ when $d(v_1) = 3$. In fact, by Lemma 5.2, if $v_1 \in V_2$ then $|V_2| \geq (4n-4)/7$; hence $v_2 \in V_2$. So by Lemma 3.1,

$$4 \leq d \leq 5, \text{ and } d = 5 \text{ only if } d(v_1) = d(v_2) = 3 \text{ and } v_1v_2 \in E(G).$$

Let $G' = G - \{v_1, v_2\}$; then $e(G') = e(G) - d$. We choose v_1, v_2 and a bisection $[R, \overline{R}]$ of G' such that

$$e(R, \overline{R}) \text{ is maximum.}$$

Without loss of generality, let $e(R) \leq e(\overline{R})$. Then $e(R) \leq (m - d - e(R, \overline{R}))/2$. Note that for $i = 1, 2$, $|N(v_i) \cap R| \leq 2$ and $|N(v_i) \cap \overline{R}| \leq 2$.

Before we proceed further, we pause to give a brief outline of the rest of our proof. First, we use $[R, \overline{R}]$ to show $v_1, v_2 \in V_2$ and $v_n \in \overline{R}$, and to provide bounds on $e(R, \overline{R})$ and $|B|$, where B is a set of vertices defined from a maximum matching M in $G - \{v_1, v_2\}$. Once this is achieved, we will forget about $[R, \overline{R}]$ and, instead, work with the maximum matching M and free vertices with respect to M. At the end, we will show that by removing at most two edges from G we obtain a complete bipartite graph with one color class consisting of two vertices, which is a contradiction as such graphs are not counterexamples to Theorem 1.1.

We now start the argument which makes use of $[R, \overline{R}]$. Let $e(R, \overline{R}) = (m-d)/2 + t$. Since $n \geq 12$ and $\delta(G) \geq 2$, G' has a matching of size at least 2; so by Lemma 6.1,

$$t \geq 1.$$

Let $\theta = \min\{|N(v_i) \cap \overline{R}| \, : \, i = 1, 2\}$. Then $\theta \leq 2$. We claim that

$$e(\overline{R}) - e(R) \geq 2, \;\; e(\overline{R}) \geq m/3 - \theta, \text{ and } e(R) \leq m/6 - t - d/2 + \theta. \tag{7.1}$$

It suffices to prove the first two inequalities as the last one follows from the second because $m - d = e(R) + e(\overline{R}) + e(R, \overline{R})$ and $e(R, \overline{R}) = (m-d)/2 + t$.

First, we prove $e(\overline{R}) - e(R) \geq 2$. Note that $e(\overline{R}) \geq m/3 - 2$; otherwise, with $S = R \cup \{v_2\}$, $[S, \overline{S}]$ is a bisection of G such that $\max\{e(S), e(\overline{S})\} \leq e(\overline{R}) + 2 < (m/3 - 2) + 2 = m/3$, a contradiction. Thus, since $d \geq 4$ and $t \geq 1$,

$$e(\overline{R}) - e(R) \geq (m/3 - 2) - (m - d - e(R, \overline{R}))/2 = m/12 + (d + 2t - 8)/4 \geq m/12 - 1/2.$$

Since $n \geq 12$ and $m \geq 3(20n - 21)/29$ (by Lemmas 5.3 and 5.4), $e(\overline{R}) - e(R) \geq 2$.

Now assume $e(\overline{R}) < m/3 - \theta$, and let $i \in \{1, 2\}$ such that $\theta = |N(v_i) \cap \overline{R}|$. Recall that $|N(v_{3-i}) \cap R| \leq 2$. Let $S = R \cup \{v_{3-i}\}$. Then $[S, \overline{S}]$ is a bisection of G, and

$$e(S) \leq e(R) + 2 \leq e(\overline{R}) \leq e(\overline{S}) = e(\overline{R}) + \theta < m/3,$$

a contradiction. This completes the proof of (7.1).

Next we show that

$$|N(z) \cap \overline{R}| > |N(z) \cap R| \text{ for some } z \in \overline{R}. \tag{7.2}$$

Suppose on the contrary $|N(u) \cap R| \geq |N(u) \cap \overline{R}|$ for all $u \in \overline{R}$. Then $2e(\overline{R}) = \sum_{u \in \overline{R}} |N(u) \cap \overline{R}| \leq \sum_{u \in \overline{R}} |N(u) \cap R| = e(R, \overline{R})$. Hence, since $e(\overline{R}) \leq m - d - e(R, \overline{R})$, $e(\overline{R}) \leq \min\{(m - d)/4 + t/2, (m - d)/2 - t\}$, which is maximized when $(m - d)/4 + t/2 = (m - d)/2 - t$; and in this case $t = (m - d)/6$, and hence

$$e(\overline{R}) \leq (m - d)/3.$$

Let $S = R \cup \{v_2\}$; then $[S, \overline{S}]$ is a bisection of G. By (7.1), $e(\overline{R}) \geq e(R) + 2$; so $e(R) \leq (m - d - e(R, \overline{R}) - 2)/2 = (m - d)/4 - t/2 - 1$. Thus, since $d \geq 4$ and $t \geq 1$,

$$e(S) \leq e(R) + 2 \leq (m - d)/4 - t/2 + 1 \leq m/4 - 1/2 < m/3.$$

Hence, $m/3 \leq e(\overline{S}) \leq e(\overline{R}) + d/2$; so

$$e(\overline{R}) \geq m/3 - d/2$$

and

$$e(R, \overline{R}) = \sum_{u \in \overline{R}} |N(u) \cap R| \geq \sum_{u \in \overline{R}} |N(u) \cap \overline{R}| = 2e(\overline{R}) \geq 2m/3 - d.$$

Moreover, $m/3 \leq e(\overline{S}) \leq e(\overline{R}) + d/2 \leq (m - d)/2 - t + d/2$; so $t \leq m/6$. Hence

$$e(R, \overline{R}) = (m - d)/2 + t \leq 2m/3 - d/2.$$

Thus,

$$\sum_{u \in \overline{R}} (|N(u) \cap R| - |N(u) \cap \overline{R}|) \leq e(R, \overline{R}) - 2e(\overline{R}) \leq \lfloor d/2 \rfloor$$

and

$$e(R) = m - d - e(R, \overline{R}) - e(\overline{R}) \leq m - d - (2m/3 - d) - (m/3 - d/2) = d/2.$$

We claim that for any $u \in \overline{R}$, $|N(u) \cap \overline{R}| \leq 2d/3$. For, suppose for some $u \in \overline{R}$, $|N(u) \cap \overline{R}| > 2d/3$. Let $S = R \cup \{u\}$. Then $[S, \overline{S}]$ is a bisection of G,

$$e(S) \leq e(R) + |N(u) \cap R| \leq \lfloor d/2 \rfloor + (n-2)/2 = (n+2)/2 < m/3$$

(since $m \geq 3(20n-21)/29$ and $n \geq 12$), and

$$e(\overline{S}) \leq e(\overline{R}) - |N(u) \cap \overline{R}| + d < (m-d)/3 - 2d/3 + d = m/3$$

(since $e(\overline{R}) \leq (m-d)/3$), a contradiction.

Suppose $v_n \in R$. Since $\Delta(G) \geq m/3 + 2$ and $d \leq 5$,

$$2 \geq \lfloor d/2 \rfloor \geq e(R) \geq d(v_n) - |N(v_n) \cap \overline{R}| - 2 \geq m/3 - (n-2)/2.$$

Hence $m \leq 3(n+2)/2$. Since $m \geq 3(20n-21)/29$ by Lemma 5.4, $n \leq 9$, a contradiction. So $v_n \in \overline{R}$. Then

$$d(v_n) \leq |N(v_n) \cap R| + |N(v_n) \cap \overline{R}| + 2 \leq 2|N(v_n) \cap \overline{R}| + \lfloor d/2 \rfloor + 2 \leq 4d/3 + \lfloor d/2 \rfloor + 2.$$

If $d(v_2) = 2$ then $d = 4$ and $9 \geq d(v_n) = \Delta(G) \geq m/3 + 2$ (so $m \leq 21$); hence, since $m \geq 3(20n-21)/29$ by Lemma 5.4, $n \leq 11$, a contradiction. So $d(v_2) = 3$. Then $\Delta(G) = d(v_n) \leq 10$, and $d(v_1) = 3$ (by Lemma 5.2). So $10 \geq m/3 + 3$ (hence $m \leq 21$), and $m \geq 3(4n-3)/5$ by Lemma 5.3. Thus $n \leq 9$, a contradiction. This completes the proof of (7.2).

By the maximality of $e(R, \overline{R})$, for any $x \in R$,

$$e(R, \overline{R}) \geq e((R \setminus \{x\}) \cup \{z\}, (\overline{R} \setminus \{z\}) \cup \{x\}).$$

Hence,

$$|N(x) \cap R| + |N(z) \cap \overline{R}| \leq |N(x) \cap \overline{R}| + |N(z) \cap R|,$$

and if $xz \in E(G)$

$$|N(x) \cap R| + |N(z) \cap \overline{R}| \leq |N(x) \cap \overline{R}| + |N(z) \cap R| - 2.$$

Therefore, for any $u \in R$,

$$|N(u) \cap \overline{R}| \geq |N(u) \cap R| + 1, \text{ and } |N(u) \cap \overline{R}| \geq |N(u) \cap R| + 3 \text{ if } uz \in E(G). \tag{7.3}$$

Let $p = e(\{v_1, v_2\}, R)$. We now show that

$$R \cap V_2 \neq \emptyset, \text{ and hence } v_1, v_2 \in V_2. \tag{7.4}$$

First, note that $\overline{R} \setminus V_2 \neq \emptyset$. For, otherwise, $\overline{R} \subseteq V_2$ and, by Lemma 3.1, $e(\overline{R}) = 0$; so $e(R) = 0$ as $e(R) \leq e(\overline{R})$. Let $S = R \cup \{v_1\}$. Then $[S, \overline{S}]$ is a bisection of G, and $\max\{e(S), e(\overline{S})\} \leq 3 < m/3$ (since $m \geq n \geq 12$), a contradiction.

If $v_1 \in V_2$ then $|V_2| \geq (4n-4)/7$ by Lemma 5.2; so $|V_2 \cap R| \geq |V_2 \setminus (\{v_1, v_2\} \cup \overline{R})| \geq (4n-4)/7 - n/2 > 0$ (since $\overline{R} \not\subseteq V_2$ and $n \geq 12$) which implies $v_2 \in V_2$. Thus, we may assume

$v_1 \in V_3$ and, hence, $v_2 \in V_3$, and $v_1v_2 \in E(G)$ by our choice of v_1 and v_2. Therefore, $d = 5$, $p \leq 4$, and $\{v_1v_2\}$ is a symmetric matching in G. Hence, $\Delta(G) \geq m/3 + 4$, and $m \leq 3(n-5)$.

By (7.3), for any $u \in R$ with $|N(u) \setminus \{v_1, v_2\}| \geq 4$, $|N(u) \cap \overline{R}| \geq 3$. Thus by Lemmas 4.3 and 4.6, $|N(u) \cap \overline{R}| \geq 3$ for all $u \in R$ except when $u \in V_3 \cup V_4$; and by Lemma 4.5 (when $u \in V_4$) or Lemmas 4.7 and 4.4 (when $u \in V_3$), there is at most one such exceptional u.

Suppose $v_n \in R$. Then, since $\Delta(G) \geq m/3 + 4$ and $p \leq 4$,

$$\begin{aligned} e(R, \overline{R}) &\geq 3(|R| - 2) + (\Delta(G) - p - e(R)) + 1 \\ &= 3|R| + \Delta(G) - e(R) - 5 - p \\ &\geq 3(n-2)/2 + (m/3 + 4) - e(R) - 9. \end{aligned}$$

By (7.1) and because $d = 5$ and $\theta \leq 2$,

$$\begin{aligned} m/3 &= e(R) + e(R, \overline{R}) + e(\overline{R}) + d - 2m/3 \\ &\geq e(R) + e(R, \overline{R}) + 5 - m/3 - 2 \\ &\geq 3(n-2)/2 - 2 \\ &= 3n/2 - 5, \end{aligned}$$

contradicting $m \leq 3(n-3)$ and $n \geq 12$.

So $v_n \in \overline{R}$. Then $e(R, \overline{R}) \geq 3(|R| - 1) + 1 = 3|R| - 2$, and thus

$$e(\overline{R}) \leq m - d - (3|R| - 2) - e(R) = m - 3|R| - e(R) - 3;$$

so by (7.1), $m/3 - \theta \leq m - 3|R| - e(R) - 3$, yielding $e(R) \leq 2m/3 - 3|R| - 1$ (as $\theta \leq 2$). Let $S = R \cup \{v_n\}$. Then $[S, \overline{S}]$ is a bisection of G. Since $m \leq 3(n-5)$, $n \geq m/3 + 5$. Hence

$$e(S) \leq e(R) + |R| \leq 2m/3 - 2|R| - 1 = 2m/3 - (n-1) < m/3$$

and

$$\begin{aligned} e(\overline{S}) &\leq e(\overline{R}) - (\Delta(G) - |R|) + (5 - p) \\ &\leq (m - 3|R| - e(R) - 3) - ((m/3 + 4) - |R|) + (5 - p) \\ &\leq 2m/3 - 2|R| - e(R) - 2 \\ &\leq 2m/3 - n \\ &< m/3. \end{aligned}$$

This is a contradiction, which completes the proof of (7.4).

By Lemma 3.1, $d = 4$ and $N(R \cap V_2) \subseteq R \cup \overline{R}$. Indeed, $N(R \cap V_2) \subseteq \overline{R}$ by (7.3). Recall that $e(V_2, V_3) = 0$ by Lemma 3.3 and that $V_4 \cup V_5 = \emptyset$ by Lemma 3.6. Therefore, by (7.3), $|N(u) \cap \overline{R}| \geq 2$ for all $u \in R$.

We claim that $v_n \in \overline{R}$. For, suppose $v_n \in R$. Then $e(R, \overline{R}) \geq 2(|R|-1) + (d(v_n) - e(R) - 2)$. Hence

$$\begin{aligned} m - 4 &= e(R) + e(\overline{R}) + e(R, \overline{R}) \\ &\geq e(R) + e(\overline{R}) + 2(|R| - 1) + (\Delta(G) - e(R) - 2) \quad \text{(as } d(v_n) = \Delta(G)) \\ &\geq (m/3 - \theta) + 2|R| + \Delta(G) - 4 \quad \text{(by (7.1))} \\ &\geq m/3 + 2|R| + m/3 - 4 \quad \text{(since } \Delta(G) \geq m/3 + 2 \text{ and } \theta \leq 2) \\ &= 2m/3 + n - 6. \end{aligned}$$

Thus, $m \geq 3(n-2)$, contradicting the fact that $m \leq 3(n-3)$.

Then

$$e(R, \overline{R}) \leq m/3 + |N(v_n) \cap R| - 4 + \theta. \tag{7.5}$$

For, otherwise, by (7.1),

$$m/3 - \theta \leq e(\overline{R}) < m - 4 - (m/3 + |N(v_n) \cap R| - 4 + \theta) - e(R).$$

Hence $e(R) + |N(v_n) \cap R| < m/3$. Let $S = R \cup \{v_n\}$. Then $e(S) = e(R) + |N(v_n) \cap R| < m/3$ and

$$\begin{aligned} e(\overline{S}) &\leq e(\overline{R}) - (d(v_n) - |N(v_n) \cap R|) + 4 - p \\ &\leq (m - 4 - e(R, \overline{R})) - (\Delta(G) - |N(v_n) \cap R|) + 4 - p \\ &< m - (m/3 + |N(v_n) \cap R| - 4 + \theta) - (m/3 + 2 - |N(v_n) \cap R|) - p \\ &= m/3 + (2 - \theta - p) \\ &\leq m/3 \quad \text{(since } \theta + p \geq 2\text{)}. \end{aligned}$$

Hence, $[S, \overline{S}]$ is a bisection in G such that $\max\{e(S), e(\overline{S})\} < m/3$, a contradiction which completes the proof of (7.5).

We now start working with maximum matchings in $G' := G - \{v_1, v_2\}$. Let $M = \{x_1y_1, x_2y_2, \ldots, x_ly_l\}$ be a maximum matching in G'. Then

$$V(G') \setminus V(M) \text{ is independent.}$$

For each $x \in V(G') \setminus V(M)$, let $F_M(x)$ denote the set of free neighbors of x with respect to M. Let $\{x_{l+1}, y_{l+1}\}, \cdots, \{x_h, y_h\}$ be pairwise disjoint pairs of vertices such that $F_M(x_j) \neq F_M(y_j)$ for $j \in \{l+1, \ldots, h\}$. Let $B(M) := V(G') \setminus \{x_j, y_j : 1 \leq j \leq h\}$. We choose v_1, v_2, M and $\{x_{l+1}, y_{l+1}\}, \cdots, \{x_h, y_h\}$ so that, subject to $v_1, v_2 \in V_2$,

- h is maximum and, then,
- $\sum_{x \in B(M)} d(x)$ is maximum.

If there is no danger of confusion we will drop the reference to M. Note that for any distinct $x, y \in B$, $F(x) = F(y)$, which is denoted as F. (The second condition above will be used to show that certain vertices are in V_2, see Claim 5.)

By Lemma 6.1 and by (7.5),

$$(m-4)/2 + (n - 2 - |B|)/4 \leq m/3 + |N(v_n) \cap R| - 4 + \theta,$$

so $|B| \geq n - 2 - 4|N(v_n) \cap R| + 2m/3 + 4(2 - \theta)$. Since $|N(v_n) \cap R| \leq |R| = (n-2)/2$, $m \geq 3(20n - 21)/29$ and $\theta \leq 2$, we have

$$|B| \geq 2m/3 - (n-2) + 4(2 - \theta) > 11n/29. \tag{7.6}$$

Since $n \geq 12$, $|B| \geq 5$.

We have finished the arguments that make use of $[R, \overline{R}]$. From now on, we will work with the pairs $\{x_i, y_i\}$, $1 \le i \le h$, and proceed with ten claims to show that by removing at most two edges from G, we obtain a complete bipartite graph with one color class consisting of two vertices.

Claim 1. If $F \neq \emptyset$ then for any $j \in \{l+1, \ldots, h\}$, $F(x_j) = F$ or $F(y_j) = F$.

For, otherwise, assume without loss of generality that $F(x_h) \neq F \neq \emptyset$ and $F(y_h) \neq F \neq \emptyset$. Let $x, y \in B$ be distinct (which exist as $|B| \ge 5$). Then M and $\{x_{l+1}, y_{l+1}\}, \ldots, \{x_{h-1}, y_{h-1}\}, \{x, x_h\}, \{y, y_h\}$ contradict the maximality of h.

Claim 2. For any distinct $x, y \in V(G') \setminus V(M)$ and for any $1 \le i \le l$, $xx_i \notin E(G')$ or $yy_i \notin E(G')$.

For, suppose $xx_i, yy_i \in E(G')$. Then $(M \setminus \{x_i y_i\}) \cup \{xx_i, yy_i\}$ is a matching in G', contradicting the maximality of M.

For any $x \in B$, let $M_x \subseteq M$ be maximal such that for each $x_i y_i \in M_x$, $xx_i, xy_i \in E(G)$; then $V(M_x) \cap F = \emptyset$, and $d_{G'}(x) = 2|M_x| + |F|$. Thus $M_x = \emptyset$ if and only if $d_{G'}(x) = |F|$. By Claim 2, $M_x \cap M_y = \emptyset$ for any distinct $x, y \in B$. Let $M' := \cup_{x \in B} M_x$. Then $|M'| = \sum_{x \in B} |M_x|$ and $|M'| \le |M| - |F|$.

Note that each vertex in F has degree at least $|B| + 1$ in G'. Since $\delta(G) = 2$, $|V_3| \le 2$ by Lemmas 4.4 and 4.7, and $|V_2| \ge (4n-4)/7 > 6$ as $n \ge 12$. Hence $V_i = \emptyset$ for $4 \le i \le 8$ by Lemmas 3.6 and 3.11. In particular, each vertex in F has degree at least 9.

Claim 3. $|F| \le 2$.

First, assume $|F| = 3$. Then, since $|V_3| \le 2$ and $V_i = \emptyset$ for $4 \le i \le 8$, at least $|B| - 2$ vertices in B each have degree at least 9 in G, and if x is such a vertex then $|M_x| = (d_{G'}(x) - |F|)/2 = (d_{G'}(x) - 3)/2 \ge (d_G(x) - 5)/2 \ge 2$. By Lemma 3.1, for each $x \in B$ at least $|V(M_x)| = |M_x|$ vertices in $N_{G'}(x) \setminus F$ have degree more than 2. So G has at least $3(|B| - 2)$ vertices of degree more than 2. Hence $|V_2| \le n - 3(|B| - 2) < 6$ by (7.6), a contradiction.

Now assume $|F| \ge 4$. We apply discharging method to G. Assign an initial charge of $d(x)$ to each $x \in V(G)$; each $x \in V(G)$ of degree at least 9 sends charge 1 to each member of $N(x) \cap V_2$; and for all $x \in V(G)$ let $\omega(x)$ denote the final charge at x. Note that for each $x \in F$, $|N(x) \cap V_2| \le d(x) - |B|$, and thus $\omega(x) \ge |B|$. For each $x \notin V_2 \cup V_3 \cup F$, $d(x) \ge 9$; so by Lemma 3.7, $\omega(x) = d(x) - |N(x) \cap V_2| > d(x) - (4d(x) - 6)/5 = (d(x) + 6)/5 \ge 3$, and hence $\omega(x) \ge 4$. Therefore, since $|V_3| \le 2$ and $\omega(x) = 3$ for $x \in V_3$,

$$\begin{aligned} 2m &= \sum_{x \in V(G)} \omega(x) \\ &\ge 4(n-6) + 4|B| + 6 \\ &= 4(|B| + n) - 18 \\ &\ge 8m/3 - 10 \quad \text{(as } |B| \ge 2m/3 - (n-2) \text{ by (7.6)).} \end{aligned}$$

Thus $m \le 15$. Since $m \ge 3(20n - 21)/29$, $n \le 8$, a contradiction (as $n \ge 12$).

Claim 4. $|F| = 2$ and $B \cap V_2 \neq \emptyset$.

Suppose $|F| \leq 1$. Then $|M_x| \geq 1$ for all $x \in B$. So by Claim 2, $|B| \leq |M'| \leq |M| \leq (n-2-|B|)/2$. Hence $|B| < n/3$, contradicting (7.6). So $|F| = 2$.

Now assume $B \cap V_2 = \emptyset$. Then, since $|F| = 2$, $|M_x| \geq 1$ for all $x \in B$; hence $|B| \leq |M'| = \sum_{x \in B} |M_x|$. Therefore, $n-2 \geq \sum_{x \in B} 2|M_x| + |F| + |B| \geq 3|B| + 2$; so $|B| \leq (n-4)/3$, contradicting (7.6). Hence we have Claim 4.

Since $|F| = 2$, every vertex in B has even degree. Without loss of generality, let $F = \{s, t\}$, and let $ss', tt' \in M$. Since M is a maximum matching in G', we see that for $l+1 \leq j \leq h$,

$$e(\{s', t'\}, \{x_j, y_j\}) = 0.$$

Claim 5. $st', ts' \in E(G)$ and $d_G(s') = d_G(t') = 2$.

First, suppose $s't \notin E(G)$ or $t's \notin E(G)$. By symmetry, let $s't \notin E(G)$. Let $x, y \in B$, and let $N = (M \setminus \{ss'\}) \cup \{sx\}$. Then N is a maximum matching in G', and $F_N(s') \neq F_N(y)$. For $l+1 \leq j \leq h$, since $e(\{s', t'\}, \{x_j, y_j\}) = 0$, we have $F_N(x_j) \neq F_N(y_j)$. Thus, N, $\{x_{l+1}, y_{l+1}\}, \ldots, \{x_h, y_h\}$, $\{s', y\}$ contradict the choice of M, $\{x_{l+1}, y_{l+1}\}, \ldots, \{x_h, y_h\}$. So $st', ts' \in E(G)$.

Next, we show $d_G(s') = d_G(t') = 2$. Otherwise, assume by symmetry that $d_G(s') > 2$. Let $x \in B \cap V_2$, and let $N = (M \setminus \{ss'\}) \cup \{sx\}$. Then N is a maximum matching in G'. Since $x \in V_2$ and, for $l+1 \leq j \leq h$, $e(\{s', t'\}, \{x_j, y_j\}) = 0$, we have $F_N(x_j) \neq F_N(y_j)$. If for some $v \in B \setminus \{x\}$, $F_N(s') \neq F_N(v)$, then N, $\{x_{l+1}, y_{l+1}\}, \ldots, \{x_h, y_h\}$, $\{s', v\}$ contradict the choice of M, $\{x_{l+1}, y_{l+1}\}, \ldots, \{x_h, y_h\}$. So $B(N) = (B \setminus \{x\}) \cup \{s'\}$. However, $\sum_{x \in B(N)} d_{G'}(x) > \sum_{x \in B} d_{G'}(x)$, again a contradiction to the choice of M and $\{x_{l+1}, y_{l+1}\}, \ldots, \{x_h, y_h\}$.

Claim 6. $N(v_i) = F$ for $i \in \{1, 2\}$.

Let $x \in B \cap V_2$. Then by the choice of v_1, v_2 and M, M is also a maximum matching in $G - \{x, v_2\}$. So v_1 is not adjacent to $B \cup \{x_{l+1}, y_{l+1}, \ldots, x_h, y_h\}$. Thus, if $v_1 t \notin E(G)$ then in $G - \{x, v_2\}$, $F_M(v_1) \neq F_M(v)$ for any $v \in B \setminus \{x\}$; so x, v_2, M, $\{x_{l+1}, y_{l+1}\}, \ldots, \{x_h, y_h\}$, $\{s, v\}$ contradict the choice of v_1, v_2, M, $\{x_{l+1}, y_{l+1}\}, \ldots, \{x_h, y_h\}$. Therefore, $v_1 t \in E(G)$. Similarly, $v_1 s, v_2 t, v_2 s \in E(G)$.

Claim 7. $|\{v \in V(G) : d(v) \geq n/3\}| \leq 9$.

By Lemma 3.2, $st \in E(G)$. So by Claims 5 and 6 and by (7.6), each vertex in $F = \{s, t\}$ has degree at least $|B| + 5 > 11n/29 + 5$. Thus, if Claim 7 fails then

$$2m > 2(n-10) + 8 \cdot (n/3) + (11n/29 + 5) + (m/3 + 2)$$

yielding $m > 3(n-3)$, a contradiction.

Claim 8. $m > 2.25n - 10.5$.

We apply a discharging argument to G. For each $v \in V(G)$ assign an initial charge of $d(v)$; and for each $v \in V_2$ and $u \in N(v)$, u sends a charge 1.25 to v. Let $\omega(v)$ denote the final charge at v, for all $v \in V(G)$. Next, we bound $\omega(v)$ for $v \in V(G)$.

Let $v \in V_k$. If $k = 2$ then $\omega(v) = 4.5$. If $k = 3$ then $\omega(v) = 3$ (as $e(V_2, V_3) = \emptyset$). When $k \geq 4$, $k \geq 9$ as $V_i = \emptyset$ for $4 \leq i \leq 8$; so by Lemma 3.7(1), $|N(v) \cap V_2| < (4k-6)/5$. Now assume $4 \leq k < n/3$. Since $|V_2| \geq (4n-4)/7$ by Lemma 5.2, $|N(v) \cap V_2| < (4k-6)/5 <$

$(4n-18)/15 < |V_2|/2$; so $|V_2 \setminus N(v)| > |N(v) \cap V_2|$. Thus by Lemma 3.7(2) (with $p = 1$), $|N(v) \cap V_2| < (k-2)/3$. Thus $\omega(v) > k - 1.25(k-2)/3 > 4.5$.

Note that if $k \geq n/3$ then a simple calculation gives $\omega(v) \geq k - 1.25(4k-6)/5 = 1.5$. Thus, if $b := |\{v \in V(G) : d(v) \geq n/3\}| \leq 6$ then, since $|V_3| \leq 2$,

$$2m = \sum_{v \in V(G)} \omega(v) > 4.5(n-b-2) + 6 + 1.5b \geq 4.5n - 21;$$

so $m > 2.25n - 10.5$. Thus we may assume $b \geq 7$. We consider the vertices of degree at least $n/3$ collectively: the total charge they send away is at most the total charge received by V_2, namely $2.5|V_2|$ which is at most $2.5(n - b - |V_3|)$ (as $|V_2| \leq n - b - |V_3|$). Hence

$$\begin{aligned}
2m &= \sum_{v \in V(G)} \omega(v) \\
&> 4.5(n-b-|V_3|) + 3|V_3| + \left(\sum_{\{v : d(v) \geq n/3\}} d(v)\right) - 2.5(n-b-|V_3|) \\
&\geq 4.5(n-b-|V_3|) + 3|V_3| + ((b-1)n/3 + m/3 + 2) - 2.5(n-b-|V_3|)) \\
&\geq 4n + |V_3| + 2 - 2b + m/3 \quad \text{(since } (b-1)n/3 \geq 2n \text{ as } b \geq 7) \\
&\geq 4n - 16 + m/3 \quad \text{(since } b \leq 9 \text{ by Claim 7).}
\end{aligned}$$

Thus $m > 2.4n - 10$, completing the proof of Claim 8.

By Claim 8 and (7.6), $|B| \geq 2m/3 - n + 2 > n/2 - 5$. Since n is even, we have that

$$|B| \geq n/2 - 4. \tag{7.7}$$

Claim 9. For any $x \in B \setminus V_2$, $F \subseteq N(v)$ for all $v \in V(M_x)$.

For, let $x \in B \setminus V_2$. Then $M_x \neq \emptyset$. Let $x_i y_i \in M_x$ and assume $F \not\subseteq N(x_i)$. Now $K := (M \setminus \{x_i y_i\}) \cup \{x y_i\}$ is a maximum matching in G', and $e(\{x_i, y_i\}, \{x_j, y_j\}) = 0$ for all $l+1 \leq j \leq h$. Let $z \in B \setminus \{x\}$. Then $F_K(x_i) \neq F_K(z)$. Hence K, $\{x_{l+1}, y_{l+1}\}, \ldots, \{x_h, y_h\}, \{x_i, z\}$ contradict the choice of M, $\{x_{l+1}, y_{l+1}\}, \ldots, \{x_h, y_h\}$.

Claim 10. $|B \setminus V_2| \leq (2.5n - 13)/71$.

For any $x \in B \setminus V_2$, since $V_i = \emptyset$ for $4 \leq i \leq 8$, we have $|M_x| \geq 4$; so $d(x) \geq 10$. Therefore, since $M_x \cap M_y = \emptyset$ for distinct $x, y \in B$, it follows from Claim 9 that G has at least $9|B \setminus V_2| + 2$ vertices of degree at least 9 (including s and t and vertices in $B \setminus V_2$). Thus, $9|B \setminus V_2| + 2 \leq n - |V_2| - |V_3|$. In fact, by Claim 9, s and t both have degree at least $|B| + 8|B \setminus V_2| + 4$. Thus, by (7.7) (see second inequality below),

$$\begin{aligned}
2m &\geq 2|V_2| + 3|V_3| + 9(n - |V_3| - |V_2| - |F|) + (|B| + 8|B \setminus V_2| + 4) + (m/3 + 2) \\
&= m/3 + 9n + |B| + 8|B \setminus V_2| - 7|V_2| - 6|V_3| - 12 \\
&\geq m/3 + 9n + (n/2 - 4) + 8|B \setminus V_2| - 7(n - |V_3| - (9|B \setminus V_2| + 2)) - 6|V_3| - 12 \\
&= m/3 + 2.5n + 71|B \setminus V_2| + |V_3| - 2 \\
&\geq m/3 + 2.5n + 71|B \setminus V_2| - 2.
\end{aligned}$$

Since $m \leq 3(n-3)$, we have $5(n-3) \geq 2.5n + 71|B \setminus V_2| - 2$; so $|B \setminus V_2| \leq (2.5n - 13)/71$.

By Claims 5 and 6, the vertices in $(B \cap V_2) \cup \{s', t', v_1, v_2\}$ have the same neighborhood in G. So by Lemma 3.8, $V_i = \emptyset$ for $i \in \{4, 5, \ldots, |B \cap V_2| + 4\}$. Note that both s and t have degree at least $|B| + 5$ (as $st \in E(G)$ by Lemma 3.2).

Suppose that G contains at least five vertices of degree greater than $|B \cap V_2| + 4$. Then

$$\begin{aligned}
2m &\geq 2(n-5) + 3(|B \cap V_2| + 4) + (|B| + 5) + (m/3 + 2) \\
&\geq 2n - 10 + 3(|B| - (2.5n - 13)/71 + 4) + (|B| + 5) + (m/3 + 2) \quad \text{(by Claim 10)} \\
&> 2n + 4|B| + m/3 - 7.5n/71 + 9 \\
&\geq 2n + 4(2m/3 - n + 2) + m/3 - 7.5n/71 + 9 \quad \text{(by (7.6))} \\
&= 3m - 2n - 7.5n/71 + 17.
\end{aligned}$$

Hence $m < 2n + 7.5n/71 - 17$, contradicting Claim 8.

Therefore, at most four vertices of G have degree greater than $|B \cap V_2| + 4$. On the other hand, for any $w \in B \setminus V_2$, since $V_i = \emptyset$ for $i \in \{4, 5, \ldots, |B \cap V_2| + 4\}$, $d(w) > |B \cap V_2| + 4$ and, by Claim 9, $d(x) > |B \cap V_2| + 4$ for all $x \in V(M_w)$. Thus, at least

$$|B \setminus V_2|(|B \cap V_2| + 4 - |F|) + |F| = |B \setminus V_2|(|B \cap V_2| + 2) + 2$$

vertices of G have degree greater than $|B \cap V_2| + 4$. Thus $|B \setminus V_2|(|B \cap V_2| + 2) \leq 2$; so $|B \setminus V_2| = 0$.

Thus, F is the neighborhood of each vertex in $B' := B \cup \{v_1, v_2, s', t'\}$. Since $|B'| = |B| + 4 \geq n/2$ (by (7.7)), it follows from Lemma 3.8 that s, t are the only two vertices of G with degree greater than 3. If $V_3 = \emptyset$ then by Lemma 3.1, $G - st$ is a complete bipartite graph with F as one of the color classes. If $V_3 \neq \emptyset$ then $|V_3| = 2$ and $e(V_3) = 1$ by Lemma 4.7; so by Lemma 3.3, $G[V_3 \cup F] = K_4$ and, hence, $G - st - E(G[V_3])$ is a complete bipartite graph with F as one of the color classes. Hence, it is straightforward to find a bisection $[S, \overline{S}]$ of G such that $\max\{e(S), e(\overline{S})\} < m/3$. This contradicts the choice of G and completes the proof of Theorem 1.1.

Acknowledgment. We thank the anonymous referees for their helpful comments. In particular, we thank the referee who suggested a fix to a missing case in the proof of Lemma 3.4 and gave a simpler proof of Lemma 3.10.

References

[1] N. Alon, B. Bollobás, M. Krivelevich and B. Sudakov, Maximum cuts and judicious partitions in graphs without short cycles, *J. Combin. Theory Ser. B* **88** (2003) 329–346.

[2] B. Bollobás and A. D. Scott, Exact bounds for judicious partitions of graphs, *Combinatorica* **19** (1999) 473–486.

[3] B. Bollobás and A. D. Scott, Problems and results on judicious partitions, *Random Struct. Alg.* **21** (2002) 414–430.

[4] B. Bollobás and A. D. Scott, Better bounds for Max Cut, in Contemporary Comb, Bolyai Soc. Math. Stud. 10, *János Bolyai Math Soc*, Budapest, 2002, pp. 185–246.

[5] B. Bollobás and A. D. Scott, Max k-cuts and judicious k-partitions, *Discrete Math.* **310** (2010) 2126–2139.

[6] B. Bollobás and A. D. Scott, Judicious partitions of bounded-degree graphs, *J Graph Theory* **46** (2004) 131–143.

[7] C. S. Edwards, Some extremal properties of bipartite graphs, *Canadian J. Math.* **25** (1973) 475–485.

[8] C. S. Edwards, An improved lower bound for the number of edges in a largest bipartite subgraph, in *Proc. 2nd Czechoslovak Symposium on Graph Theory*, Prague (1975) 167–181.

[9] U. Feige and M. Langberg, The RPR^2 rounding technique for semidefinite programs, *J, Algorithms* **60** (2006) 1–23.

[10] M. X. Goemans and D. P. Williamson, Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming, *J. ACM* **42** (1995) 1115–1145.

[11] R. M. Karp, Reducibility among combinatorial problems, in *Complexity of Computer Computations* (R. Miller and J. Thatcher, eds.) Plenum Press, New York (1972) 85–103.

[12] S. Khot, G. Kindler, E. Mossel and R, O'Donnell, Optimal inapproximability results for MAX-CUT and other 2-variable CSPs? *SIAM J. Computing* **37** (2007) 319–357.

[13] C. Lee, P. S. Loh and B. Sudakov, Bisections of graphs, *J. Combin. Theory* Ser. B **103** (2013) 599–629.

[14] S. Poljak and Z. Tuza, Maximum cuts and largest bipartite subgraphs, in *Combinatorial Optimization* (W. Cook, L. Lovász, and P. D. Seymour eds.) DIMACS series in Discrete Mathematics and Theoretical Computer Science, **20** Amer. Math. Soc. Providence, RI.

[15] T. D. Porter, On a bottleneck bipartition conjecture of Erdős, *Combinatorica* **12** (1992) 317–321.

[16] A. Scott, Judicious partitions and related problems, In *Surveys in Combinatorics*, The proceedings of the Twentieth British Combinatorial Conference, Durham, UK, July 2005.

[17] F. Shahrokhi and L. A. Székely, The complexity of the bottleneck graph bipartition problem, *J. Combin. Math. Combin. Comp.* **15** (1994) 221–226.

[18] B. Xu, J. Yan and X. Yu, Balanced judicious partitions of graphs, *J. Graph Theory* **63** (2010) 210–225.

[19] B. Xu, J. Yan and X. Yu, A note on balanced bipartitions, *Discrete Math.* **310** (2010) 2613–2617.

[20] B. Xu and X. Yu, Judicious k-partitions of graphs, *J. Combin. Theory* Ser. B **99** (2009) 324–337.

[21] B. Xu and X. Yu, Better bounds for k-partitions of graphs, *Combin., Probab. and Comp.* **20** (2011) 631–640.

[22] M. Yannakakis, Node- and edge-deletion NP-complete problems, *STOC* (1978) 253–264.