Hamilton circles in infinite planar graphs

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Abstract

A circle in a graph G is a homeomorphic image of the unit circle in the Freudenthal compactification of G, a topological space formed from G and the ends of G. Bruhn conjectured that every locally finite 4-connected planar graph G admits a Hamilton circle, a circle containing all points in the Freudenthal compactification of G that are vertices and ends of G. We prove this conjecture for graphs with no *dividing* cycles. In a plane graph, a cycle C is said to be dividing if each closed region of the plane bounded by C contains infinitely many vertices.

Keywords: ray; double ray; dividing cycle; Tutte subgraph; end; Hamilton circle
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1 Introduction

Whitney [19] proved that every finite 4-connected planar triangulation contains a Hamilton cycle. This result was generalized by Tutte to the following

Theorem 1.1 (*Tutte [18]*). *Every finite* 4*-connected planar graph contains a Hamilton cycle.*

Thomassen [16] further generalized Theorem 1.1 by showing that every finite 4-connected planar graph is Hamilton connected. In this paper we consider a possible generalization of Theorem 1.1 to infinite graphs, conjectured by Bruhn (see [5]).

A *ray* is a graph isomorphic to the graph with vertex set $\{v_i : i = 1, 2, \ldots\}$ and edge set $\{v_iv_{i+1} : i = 1, 2, \ldots\}$. A *double ray* is a graph isomorphic to the graph with vertex set $\{v_i : i = \ldots, -2, -1, 0, 1, 2, \ldots\}$ and edge set $\{v_iv_{i+1} : i = \ldots, -2, -1, 0, 1, 2, \ldots\}$. Subrays of rays or double rays are their *tails*.

In an attempt to generalize Theorem 1.1 to infinite graphs, Nash-Williams ([11, 12], also see [17]) conjectured that an infinite 4-connected planar graph contains a spanning ray if, and only if, it is 2-indivisible. (A graph is said to be *k-indivisible* if the deletion of any finite set of vertices results in at most $k - 1$ infinite components.) This conjecture is verified in [4]. Nash-Williams also conjectured that an infinite 4-connected planar graph contains a spanning double ray if, and only if, it is 3-indivisible. This conjecture is proved in [20–24]. The indivisibility conditions in Nash-Williams' conjectures (compared with Tutte's theorem) indicate that spanning rays and spanning double rays are not, in a sense, the right infinite version of a Hamilton cycle.

Note that a locally finite graph is k-indivisible if, and only if, it has at most $k - 1$ ends. An *end* of a graph is an equivalence class of rays; two rays are *equivalent* if there are infinitely many vertex disjoint paths between them. For example, a ray has just one end, while a double ray has exactly two ends.

An end may be thought of as a "point at infinity" to which its rays "converge". Viewing an infinite locally finite graph as a 1-complex and compactifying it by adding its ends as points at infinity, one obtains a topological space which is compact and Hausdorff. (See [5] for the definition of basic open neighborhoods.) This space is called the *Freudenthal compactification* of the graph. For a locally finite graph G, we follow [5] to use $|G|$ to denote the Freudenthal compactification of G. A *circle* in $|G|$ is a homeomorphic image of the unit circle, see [5–7]. Under this definition, finite cycles in G are circles in $|G|$. We shall sometimes abuse the notation by also speaking of a circle in G. An *arc* in $|G|$ is a homeomorphic image of the unit interval.

A *Hamilton circle* in $|G|$ is a circle in $|G|$ which contains all vertices (and hence, all ends) of G. It is shown (recently) in [9] that the square of every locally finite graph admits a Hamilton circle, confirming a conjecture of Diestel [5] and generalizing the result in [8] that the square of every finite 2-connected graph is hamiltonian.

The following conjecture of Bruhn (reported by Diestel in [5]) would generalize Theorem 1.1 to infinite graphs.

Conjecture 1.2 *Every locally finite* 4*-connected planar graph admits a Hamilton circle.*

Recently, Bruhn and Yu [3] proved Conjecture 1.2 for 6-connected graphs with finitely many ends. The next result, together with Theorem 1.1, shows that Conjecture 1.2 holds for graphs with at most one end.

Theorem 1.3 (*Yu [20, 21]*). *Every infinite locally finite* 4*-connected planar graph with exactly one end admits a Hamilton circle.*

Infinite locally finite graphs with one end contain no dividing cycles. A finite cycle C in a plane graph G is said to be *dividing* if each closed region of the plane bounded by C contains infinitely many vertices of G. The main purpose of this paper is to establish Conjecture 1.2 for graphs with no dividing cycles.

Theorem 1.4 *Let G be an infinite locally finite* 4*-connected plane graph. If G has no dividing cycle, then G admits a Hamilton circle.*

In order to describe the idea of our proof of Theorem 1.4, we need to define bridges and Tutte subgraphs. Let G be a graph (finite or infinite) and H be a subgraph (finite or infinite) of G. An *H-bridge* of G is a (finite or infinite) subgraph of G which is induced by either (1) an edge of $E(G) - E(H)$ with both incident vertices on H or (2) the edges contained in a component of $G - V(H)$ and the edges from this component to H. If B is an H-bridge of G, then the vertices in $V(H \cap B)$ are called the *attachments* of B (on H). We say that H is a *Tutte subgraph* of G if every H-bridge of G is finite and has at most three attachments on H. For any subgraph (finite or infinite) C of G, we say that H is a *C-Tutte* subgraph of G if H is a Tutte subgraph of G and every H-bridge of G containing an edge of C has at most two attachments.

Our approach to proving Theorem 1.4 is as follows; a more detailed overview is given in the next section. We work with 2-connected (and almost 4-connected) graphs, avoiding the difficulty of maintaining 4-connectivity. In such a graph G, we find a collection S of double rays such that the union of the double rays in S is a Tutte subgraph of G whose closure is a circle in $|G|$. (Therefore, when G is 4-connected, that union is a spanning subgraph of G whose closure in $|G|$ gives a Hamilton circle.) For that purpose, we need to find a sequence of collections of finite paths such that these finite paths "converge" to the desired double rays (by applying a variation of the König Infinity Lemma). To achieve this convergence, such finite paths need to move "forward" towards the ends of G. This will be made precise by describing a structure of G and by defining forward paths with respect to that structure (in section 5). Two main lemmas will be given in section 4, for the purpose of extending Tutte subgraphs in the process of finding forward paths.

We consider simple graphs only. Let P be a path and x, y be distinct vertices of P; then xPy denotes the finite subpath of P between x and y. For a finite cycle C in a plane graph and for distinct vertices x, y of C, we use xCy to denote the subpath of C from x to y in clockwise order. If G is a finite 2-connected plane graph, then the boundary of each face of G is a cycle (called *facial cycle* of G), and the cycle of G bounding its infinite face is called the *outer cycle* of G.

Throughout the rest of the paper, a graph may be finite or infinite unless it is clear from the context. For convenience, we use the notation $A := B$ to rename B with A. Let G be a graph, for $H \subseteq G$ and $S \subseteq V(G) \cup E(G)$, we use $H + S$ to denote the graph with vertex set $V(H) \cup (S \cap V(G))$ and edge set $E(H) \cup \{uv \in S \cap E(G) : \{u, v\} \subseteq (S \cap V(G)) \cup V(H)\}$. When $S \subseteq V(H)$, we use $H - S$ to denote the graph obtained from H by deleting S and all edges of H incident with S. If $S = \{s\}$, we simply write $H + s$ and $H - s$ instead of $H + S$ and $H - S$, respectively. A *block* in a graph G is either a maximal 2-connected subgraph of G or a subgraph of G induced by a cut edge of G.

2 Overview of the proof

Since the proof of Theorem 1.4 is quite complex, we give an overview here. By a face in a plane graph G, we shall always mean a connected component of the complement of the topological closure of G in the plane. If the boundary of a face is a finite cycle, say C, then we say that C is a *facial cycle*. Note that our graphs G will have nice embeddings (see section 3) and we shall mainly work with faces of finite subgraphs of G. However, we mention that for any 2-connected locally finite plane graph G, it follows from [13] (also see [1]) that the face boundaries of G are circles in $|G|$.

For inductive purposes we shall prove the following stronger result: Let G be an infinite locally finite 4-connected plane graph without dividing cycles, C be a facial cycle of G, and $e \in E(C)$; then $|G|$ has a Hamilton circle containing e. This is done in a "constructive" way: We find a spanning subgraph T of G such that $e \in E(T)$ and the closure of T in $|G|$ is a Hamilton circle. Such T consists of disjoint double rays.

To facilitate our discussion, we first show that G has a "nice embedding" in the plane such that C is a facial cycle, and the closed disc of the plane bounded by any finite cycle of G contains only finitely many vertices and edges of G. This is done in section 3.

To find the subgraph T, we let H denote the infinite block of $G - V(C)$ (which is well defined because G is 4-connected and planar), and let D denote the facial cycle of H that bounds the face of H containing C (which is a disc since G is nicely embedded). We wish to find a subgraph T' of H that can be extended to the desired subgraph T of G. A problem arises, namely, H need not be 4-connected. However, H is almost 4-connected, in the sense that if S is a cut of H of size at most 3 then every component of $H - S$ contains a vertex of D; and we say that H is $(4, D)$-connected (formally defined later). So instead of requiring that G be 4-connected, we shall assume that G is $(4, C)$-connected.

Because of this relaxation of connectivity, T is not necessarily spanning. As in the finite cases (such as the proof of Theorem 1.1) one instead requires that T be a Tutte subgraph of G (therefore, if G is 4-connected then T is a spanning subgraph of G.) In fact, for inductive purpose, we further require that T be a C-Tutte subgraph of G. In section 3, we include several known results about Tutte subgraphs of finite graphs.

Thus we shall prove the even stronger result (Theorem 5.5 in section 5): Let G be a plane graph and C be a facial cycle of G such that G is $(4, C)$-connected, and let $e \in E(C)$; then G contains a C-Tutte subgraph T such that $e \in E(T)$ and the closure of T in $|G|$ is a circle.

We shall assume that G has at least two ends; for, otherwise, Theorem 1.4 follows from Theorem 1.1 and Theorem 1.3. The proof of Theorem 5.5 then proceeds as follows.

We work with a nice embedding of G in which C is a facial cycle. Suppose there is a cycle that is disjoint from C and bounds a disc containing C. Then $G - V(C)$ has a unique infinite block, say H. Let D denote the facial cycle of H bounding the face of H containing C. We wish to find a D-Tutte subgraph of H that can be extended to the desired Tutte subgraph T of G. This, however, is not possible without adding additional properties on the Tutte subgraph of H. So we construct a new graph G_1 from H by adding to it an appropriate vertex v of C and edges from v to certain vertices on D, and the resulting graph G_1 has a facial cycle C_1 (which bounds the face of G_1 containing C) so that G_1 is $(4, C_1)$-connected. Moreover, certain C_1-Tutte subgraphs of G_1 can be extended to the desired Tutte subgraph T of G. This step of the proof is taken care of by a variation of a lemma proved in [20] (stated as Lemma 3.3 in this paper).

After repeating this argument (in the proof of Theorem 5.5 in section 5), we arrive at a plane graph G_n with a facial cycle C_n and an edge $e_n \in E(C_n)$ such that G_n is $(4, C_n)$-connected, and if G_n has a C_n-Tutte subgraph P_n containing e_n then G has the desired C-Tutte subgraph T containing e. Moreover, since G has at least two ends and G is nicely embedded, we may assume that no finite cycle in G_n disjoint from C_n bounds a disc that contains C. (Note that when no finite cycle in G disjoint from C bounds a disc containing C, this argument is not necessary and we simply let $G_n = G$ and $C_n = C$.) See Figure 18 in section 5 for an illustration of G_n and C_n.

To find the desired Tutte subgraph P_n of G_n, we need to split the graph G_n to certain subgraphs. This is done in section 5 (in the proof of Theorem 5.5). In Figure 18 such subgraphs are labeled as B_1, B_2, B_3 and so on. (See section 4 and Figure 1 for the definition of such subgraphs.) Each B_i has a spine H_i which, in the case of those obtained from G_n, is a subpath of C_n between two vertices x_i and y_i such that $B_i - V(H_i)$ is connected. To find the desired P_n, we need to find an H_i-Tutte subgraph T_i of B_i such that the closure of T_i in $|B_i|$ is an arc between x_i and y_i. This is proved in Lemma 5.4 in section 5. (Each T_i consists of a ray from x_i, a ray from y_i, and double rays.)

Therefore, the proof of Theorem 5.5 reduces to the problem of finding a Tutte subgraph (of each B_i) whose closure in $|B_i|$ is an arc between x_i and y_i. To solve this problem, we need to produce a layered structure in a similar fashion as we described above for producing the graph G_n from G, and prove two lemmas similar to Lemma 3.3. Basically, from each B_i we produce graphs $B_1^2, \ldots, B_{k_2}^2$ such that certain Tutte subgraphs of $\bigcup B_s^2$ can be extended to the desired Tutte subgraph of B_i. This is taken care of by Lemma 4.2. However, to produce those Tutte subgraphs of $\bigcup B_s^2$, we need to construct from $\bigcup B_s^2$ graphs $B_1^3, \ldots, B_{k_3}^3$ so that certain Tutte subgraphs of $\bigcup B_t^3$ can be extended to the desired Tutte subgraphs of $\bigcup B_s^2$. This is done by applying Lemma 4.1. We may think of the spines of B_i as forming layer 1, the spines of B_s^2 forming layer 2, the spines of B_t^3 forming layer 3, and so on. This process is repeated according to the parity of layers: Lemma 4.2 applies to odd layers (to extend disjoint double rays so that the closure of the extension is an arc) and Lemma 4.1 applies to even layers (to extend disjoint arcs to a circle). We find a sequence of finite Tutte subgraphs (consisting of disjoint paths) in appropriate finite graphs. We then use a variation of the König Infinity Lemma to show that such sequence has a subsequence that converges to the desired arc. To ensure the convergence, every path in our finite Tutte subgraphs must move towards the ends of the graph. For this purpose, we define a "forward" notion based on the layered structure. This is done in section 5.

3 Tutte paths and nice embeddings

We begin with two results on Tutte paths in finite planar graphs. The first is a result of Thomassen which is used to prove that finite 4-connected planar graphs are Hamilton connected.

Lemma 3.1 (*Thomassen [16]*). *Let G be a finite 2-connected plane graph with a facial cycle C. Assume $u \in V(C)$, $e \in E(C)$, and $v \in V(G) - \{u\}$. Then G contains a C-Tutte path P from u to v such that $e \in E(P)$.*

The next result is proved in [14], which is used to prove that every finite 4-connected projective-planar graph contains a Hamilton cycle.

Lemma 3.2 (*Thomas and Yu [14]*). *Let G be a finite 2-connected plane graph with a facial cycle C. Let $u, v \in V(C)$ be distinct, let $e, f \in E(C)$, and assume that u, v, e, f occur on C in clockwise order. Then G contains a vCu-Tutte path P from u to v such that $\{e, f\} \subseteq E(P)$.*

It is easy to see that the edges e and f in the above lemmas can be replaced with vertices. Hence, when these lemmas are applied, we allow e or f or both to be vertices.

By the Jordan curve theorem, any finite cycle C in an infinite plane graph G divides the plane into two closed regions (whose intersection is C). If exactly one of these two closed regions, say $\mathcal{R}$, contains only finitely many vertices and edges of G, then we use $I_G(C)$ to denote the subgraph of G consisting of the vertices and edges of G contained in $\mathcal{R}$. Note that $I_G(C)$ is a finite subgraph of G. If there is no confusion, we use $I(C)$ instead of $I_G(C)$. Clearly, $C \subseteq I(C)$, and if $I(C) = C$ then C is a facial cycle. Moreover, a finite cycle C is dividing if, and only if, $I(C)$ is not defined.

Let G be a graph and let C be a subgraph of G. We say that G is $(4, C)$*-connected* if G is 2-connected and, for any cut set $X \subseteq G$ with $|X| \leq 3$, every component of $G - X$ contains a vertex of C. Thus, if G is 4-connected, then G is also $(4, C)$-connected.

The following result is essentially the same as Theorem 2.1 in [20] where it is used to prove Theorem 1.3 in this paper. We shall use it in the proof of Theorem 5.5; the reader may want to skip reading it until then.

Lemma 3.3 (*Yu [20]*). *Let G be an infinite 2-connected plane graph, let C be a facial cycle of G, and let uv be an edge of C. Assume that G is $(4, C)$-connected and there is a finite cycle C^* in G such that $C \cap C^* = \emptyset$, $I_G(C^*)$ is defined, and $C \subseteq I_G(C^*)$. Then, there exist a 2-connected infinite plane graph G', a facial cycle C' of G', and a path $u'v'w'$ in C' such that*

(1) *G' is $(4, C')$-connected and $G' - v'$ is 2-connected;*

(2) *$G' - \{u'v', v'w'\} \subseteq G$, and no edge of G joins a vertex of $G' - V(C')$ to a vertex of $G - V(G')$;*

(3) *$(G + \{u'v', v'w'\}) - (V(G') - V(C'))$ is finite, and has a plane representation in which C and C' are facial cycles;*

(4) *$v' \neq v$ and $(C' - v') \cap C = \emptyset$;*

(5) *for any subgraph (finite or infinite) X of G' with $C' \subseteq X$, and for any C'-Tutte subgraph (finite or infinite) P' of X containing $u'v'$ such that v' has exactly one neighbor, say w, in $P' - u'$, there is a C-Tutte subgraph (finite or infinite) P of $G - (V(G') - V(X))$ through uv such that $P' - v' \subseteq P$, $P - V(P' - v')$ is a w-u' path, and, for any $z \in V(P) - V(P')$, either $z \notin V(X)$ or $z \in V(Z)$ for some P'-bridge Z of X containing an edge of C'.*

Remark. The difference between Lemma 3.3 here and Theorem 2.1 in [20] is that P' and P in (5) above are Tutte subgraphs (while they are Tutte paths in [20]). For the same proof of Theorem 2.1 in [20] to apply here as well, we need the condition in (5) that v' has exactly one neighbor, say w, in $P' - u'$. As a consequence of this new condition, $P - V(P' - v')$ is a w-u' path. (Similar, but more detailed, constructions and arguments are given in the proofs of Lemma 4.1 and Lemma 4.2.)

It will be convenient to work with certain plane representations of planar graphs. (Although it is not needed here, we nevertheless note that if a plane graph G has no dividing cycles, then the embedding of G may be modified to give a VAP-free embedding of G. Here VAP stands for vertex accumulation point, see [10,15].) We say that an infinite plane graph G is *nicely embedded* or is a *nice (plane) embedding* if, for any finite cycle C in G for which $I(C)$ is defined, $I(C)$ is contained in the closed disc bounded by C. The following result is Lemma 2.1 in [21].

Lemma 3.4 (*Yu [21]*). *Let G be an infinite plane graph with a sequence of finite cycles $(D_1, D_2, \ldots)$ such that $I(D_i)$ is defined for all $i \geq 1$, $I(D_i) \subseteq I(D_{i+1})$ for all $i \geq 1$, and $G = \bigcup_{i\geq 1} I(D_i)$. Then for any facial cycle C of G, G has a nice embedding in which C is a facial cycle.*

The next result describes (to some extent) the structure of infinite plane graphs with no dividing cycles. It is similar to Lemma 2.3 in [21].

Theorem 3.5 *Let G be an infinite 2-connected plane graph with no dividing cycles, let C be a facial cycle of G, and assume that G is $(4, C)$-connected. Then there is an infinite sequence $(D_1, D_2, \ldots)$ of finite cycles in G such that $C \subseteq I(D_1)$ and the following properties hold:*

(1) *for each $i \geq 1$, $I(D_i) \subseteq I(D_{i+1})$, and $D_i \cap D_{i+1}$ is minimal among all subgraphs $D_i \cap D^*$ arising from finite cycles D^* in G such that $I(D_i) \subseteq I(D^*)$;*

(2) *for each $i \geq 1$, G has no finite $I(D_i)$-bridge;*

(3) *for each $i \geq 1$, $D_i \cap D_{i+1} \subseteq D_{i+1} \cap D_{i+2}$;*

(4) $\bigcup_{i\geq 1} I(D_i) = G$.

Note that the graph G in Lemma 3.4 and Theorem 3.5 need not be locally finite. Also note that when Theorem 3.5 is applied, we only need $I(D_i) \subseteq I(D_{i+1})$ (for all $i \geq 1$) and properties (3) and (4).

The difference between Theorem 3.5 above and Lemma 2.3 in [21] is that the graph G in Theorem 3.5 has no dividing cycles, while the graph G in Lemma 2.3 in [21] is required to be cohesive. (A graph is *cohesive* if it is 2-indivisible and the deletion of finitely many vertices results in only finitely many components.) Because the graph G in Theorem 3.5 is planar and $(4, C)$-connected, the deletion of finitely many vertices results in only finitely many components. Also, the sole purpose of the 2-indivisibility condition used in the proof of Lemma 2.3 in [21] is to ensure that $I(D)$ is defined for every finite cycle D in G. Since the graph G in Theorem 3.5 above has no dividing cycles, $I(D)$ is defined for every finite cycle D in G. Hence with slight modification, the proof of Lemma 2.3 in [21] also gives a proof of Theorem 3.5 above.

By Theorem 3.5 and Lemma 3.4, we have the following.

Corollary 3.6 *Let G be an infinite 2-connected plane graph with no dividing cycles, let C be a facial cycle of G, and assume that G is $(4, C)$-connected. Then G has a nice embedding in which C is a facial cycle.*

4 Tutte subgraphs and 4-tuples

The aim of this section is to prove two lemmas for extending Tutte subgraphs according to the parity of a layered structure. These lemmas will be used in the next section to find certain finite paths that converge to double rays. Both lemmas are similar in flavor to Lemma 3.3.

First we define 4-tuples. We say that (G, H, x, y) is *4-tuple* if G is an infinite locally finite plane graph with no dividing cycles such that

(i) G is nicely embedded in the plane,

(ii) G is 2-connected,

(iii) there is a double ray F in G such that the vertices and edges of F are incident with a common face of G, and

(iv) x, y are distinct vertices on F, and $H = xFy$.

F is said to be the *frame* of the 4-tuple, and H is said to be the *spine* of the 4-tuple. In Figure 1, F is represented by the darkened double ray. We also say that the 4-tuple (G, H, x, y) is *associated* with G.

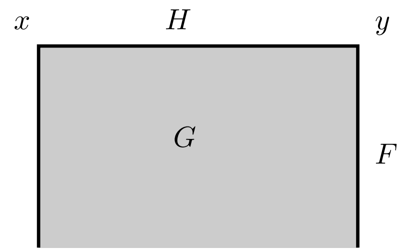


Figure 1: 4-tuple (G, H, xy).

We now state and prove the first lemma of this section. Basically, the lemma says that given a 4-tuple associated with a graph G, there exist 4-tuples associated with graphs $B_1, \ldots, B_k$ (which are subgraphs of G) such that a certain Tutte subgraph T' of $\bigcup_{i=1}^k B_i$ can be extended to a Tutte subgraph T of G. The statement of this lemma is more general, so that it can be applied to finite subgraphs of G (see condition (4) in the lemma). When Lemma 4.1 is applied later, the closure of T' will be a disjoint union of arcs, and the closure of T will be a circle.

Lemma 4.1 *Let (G, H, x, y) be a 4-tuple such that G is $(4, H)$-connected. Then for any $e \in E(H)$, there exist 4-tuples (B_i, H_i, x_i, y_i) $(1 \leq i \leq k)$ and there exist $e_i \in E(H_i)$ such that*

(1) *B_i is an induced subgraph of $G - V(H)$, and B_i is $(4, H_i)$-connected;*

(2) *no edge of G joins a vertex of $B_i - V(H_i)$ to a vertex of $G - V(B_i)$;*

(3) *$G - V(B)$ is finite (where $B := \bigcup_{i=1}^k B_i$), and $B_i \cap B_l = \emptyset$ for $1 \leq i < l \leq k$ unless $l = i+1$, $y_i = x_{i+1}$, and $V(B_i \cap B_{i+1}) = \{y_i = x_{i+1}\}$;*

(4) *for any subgraph (finite or infinite) X of B containing $H' := \bigcup_{i=1}^k H_i$, and for any H'-Tutte subgraph (finite or infinite) T' of X containing $\{e_i, x_i, y_i : 1 \leq i \leq k\}$, there exists an H-Tutte subgraph (finite or infinite) T of $G - (V(B) - V(X))$ containing $\{e, x, y\}$ such that*

  (i) *$T' \subseteq T$,*

  (ii) *for any $z \in V(T) - V(T')$, either $z \notin V(X)$, or $z \in V(Z)$ for some T'-bridge Z of X containing an edge of H', and*

  (iii) *$T - (V(T') - \{x_i, y_i : 1 \leq i \leq k\})$ is the disjoint union of y_i-x_{i+1} paths (one for each $1 \leq i \leq k$), where $x_{k+1} := x_1$.*

Proof. Let F denote the frame of (G, H, x, y), and let F_x, F_y denote the H-bridges of F containing x, y, respectively. Since G is 2-connected and by planarity, there exists a path H^* from some

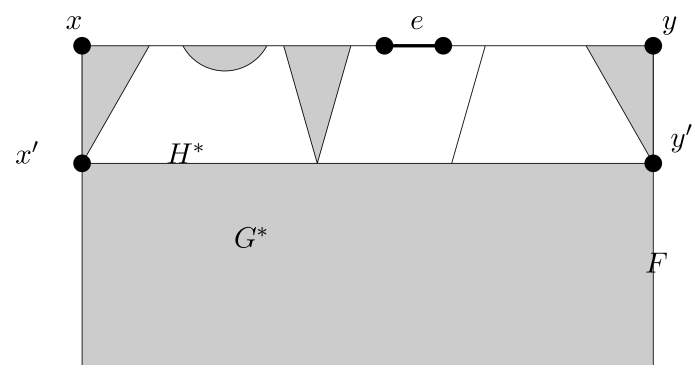


Figure 2: G and G^*.

$x' \in V(F_x - x)$ to some $y' \in V(F_y - y)$ such that $H^* \cap H = \emptyset$. Note that $C := H \cup yFy' \cup H^* \cup x'Fx$ is a cycle in G. Since G has no dividing cycles, $I_G(C)$ is defined. We may choose H^*, x' and y' so that $I_G(C)$ is minimal. See Figure 2.

Let $G^* := G - (V(I_G(C)) - V(H^*))$. By planarity, all attachments on G^* of $(H \cup G^*)$-bridges of G are contained in $V(H^*)$. By the minimality of $I_G(C)$, any $(H \cup G^*)$-bridge of G has at most one attachment on H^*. Therefore, since G is $(4, H)$-connected and by planarity, any $(H \cup G^*)$-bridge of G with only one attachment on H must be induced by a single edge.

Because G is 2-connected and planar, all cut vertices of G^* are contained in H^*, neither x' nor y' is a cut vertex of G^*, and each block of G^* contains an edge of H^*. So by planarity $F_x - V(xFx' - x')$ and $F_y - V(yFy' - y')$ each are contained in an infinite block of G^*. Since G is locally finite, G^* has only finitely many blocks. Let B_i, $1 \le i \le k$, be the infinite blocks of G^* and, for each $1 \le i \le k$, let x_i, y_i be distinct vertices of B_i such that $x_1 = x'$ and $y_k = y'$, $x_1, y_1, \ldots, x_k, y_k$ occur on H^* in order, and $x_2, \ldots, x_k, y_1, \ldots, y_{k-1}$ are cutvertices of G^*. Let $H_i := x_iH^*y_i$. Then for each $i \in \{1, \ldots, k\}$, (B_i, H_i, x_i, y_i) is a 4-tuple. For $1 \le i \le k-1$ with $y_i \ne x_{i+1}$, let Y_i denote the finite $\{y_i, x_{i+1}\}$-bridge of G^*. (Because G is $(4, H)$-connected, Y_i is uniquely defined and contains $y_iH^*x_{i+1}$.)

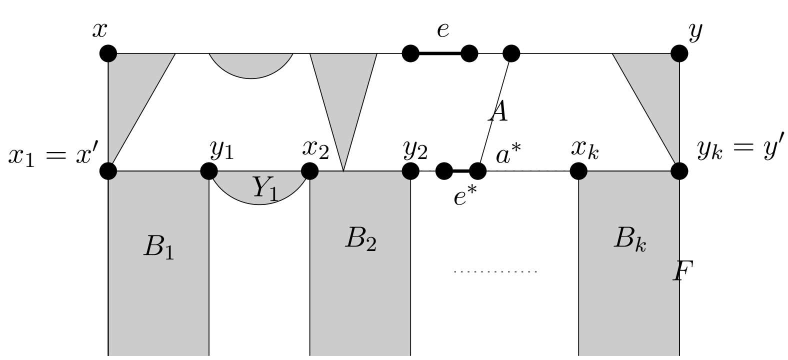


Figure 3: The infinite blocks $B_1, B_2, \ldots, B_k$ of G^*.

Clearly, each B_i is an induced subgraph of $G - V(H)$. Since G is $(4, H)$-connected and by planarity, each B_i is $(4, H_i)$-connected. So (1) holds. From the constructions of G^* and

(B_i, H_i, x_i, y_i), we see that (2) follows from planarity. Note that $G - V(\bigcup_{i=1}^k B_i)$ is finite, because it is contained in $I_G(C) \cup (\bigcup_{i=1}^{k-1} Y_i)$. Also note that if $1 \leq i < l \leq k$, then $B_i \cap B_l = \emptyset$ unless $l = i+1$, $y_i = x_{i+1}$, and $V(B_i \cap B_{i+1}) = \{y_i = x_{i+1}\}$. So (3) holds.

We now prove (4). We begin by defining $e^* \in E(H^*)$ and $e_i \in E(H_i)$ $(1 \leq i \leq k)$. Choose a path A from $a^* \in V(H^*)$ to the component of $H - e$ containing y such that A is internally disjoint from $H \cup G^*$ and, subject to this, $x_1 H^* a^*$ is minimal. See Figure 3. Thus, any path from $x_1 H^* a^* - a^*$ to H and internally disjoint from $H \cup G^*$ must intersect the component of $H - e$ containing x. Choose the edge e^* from $E(H^*)$ so that e^* is incident with a^*. (This is to ensure that when we later extend T' to T, we can require $e \in E(T)$.) For each $1 \leq i \leq k$, let $e_i = e^*$ if $e^* \in E(H_i)$; and otherwise let $e_i \in E(H_i)$ be arbitrary.

To prove (4), let X be a subgraph (finite or infinite) of $B := \bigcup_{i=1}^k B_i$ containing $H' := \bigcup_{i=1}^k H_i$, and assume that T' is an H'-Tutte subgraph (finite or infinite) of X containing $\{e_i, x_i, y_i : 1 \leq i \leq k\}$. We proceed to find the disjoint y_i-x_{i+1} paths (one for each $1 \leq i \leq k$, and $x_{k+1} = x_1$) whose union with T' gives rise to T.

For each $1 \leq i \leq k-1$, we find a y_i-x_{i+1} path P_i in Y_i such that P_i is a $y_i H^* x_{i+1}$-Tutte path in Y_i, and $e^* \in E(P_i)$ whenever $e^* \in E(Y_i)$. If $|V(y_i H^* x_{i+1})| \leq 3$, then $Y_i = y_i H^* x_{i+1}$ (since G^* is $(4, H^*)$-connected) and $P_i := Y_i$ is the desired path. Now assume $|V(y_i H^* x_{i+1})| \geq 4$. Then $Y_i + y_i x_{i+1}$ is 2-connected; and we may assume that $Y_i + y_i x_{i+1}$ is a plane graph in which $y_i H^* x_{i+1} + y_i x_{i+1}$ is a facial cycle. By Lemma 3.1, there is a $(y_i H^* x_{i+1} + y_i x_{i+1})$-Tutte path P_i from y_i to x_{i+1} in $Y_i + y_i x_{i+1}$ through an edge of $y_i H^* x_{i+1}$ (chosen to be e^* when $e^* \in E(Y_i)$). Then P_i is the desired path.

Next we find the y_k-x_1 path P_k containing $\{x, y, e\}$. For convenience, let $T^* := T' \cup (\bigcup_{i=1}^{k-1} P_i)$ and $X^* := X \cup (\bigcup_{i=1}^{k-1} Y_i)$. See Figure 4 for an illustration. Note that $e^* \in E(T^*)$, and T^* is an H^*-Tutte subgraph of X^* containing $\{x_1, y_k, e^*\}$.

Let W denote the set of attachments on H^* of $(H \cup G^*)$-bridges of G. We define an equivalence relation $\sim$ on W as follows. For any $w, w' \in W$, $w \sim w'$ if $w = w'$, or $\{w, w'\} \subseteq V(D) - V(T^*)$ for some T^*-bridge D of X^*. Let $W_1, \ldots, W_m$ be the equivalence classes of W with respect to $\sim$. Then either $|W_i| = 1$ and $W_i \subseteq V(T^*)$ (in which case, let $D_i := W_i$), or there exists a T^*-bridge D_i of X^* such that $W_i \subseteq V(D_i) - V(T^* \cap D_i)$. By planarity, we may assume that $W_1, \ldots, W_m$ occur on H^* in order, with $W_1 = \{x_1\}$ and $W_m = \{y_k\}$ (because $x_1, y_k \in V(T^*)$).

For each $1 \leq i \leq m$, let $s_i, t_i \in V(H)$ with $s_i H t_i$ maximal such that (a) x, s_i, t_i, y occur on H in order, and (b) there exist $w_s, w_t \in W_i$ such that $\{s_i, w_s\}$ and $\{t_i, w_t\}$ each are contained in an $(H \cup G^*)$-bridge of G. By planarity, $s_1 = x$, $t_m = y$, and $s_1, t_1, s_2, t_2, \ldots, s_m, t_m$ occur on H in order. See Figure 4.

For each $1 \leq i \leq m-1$, let I_i denote the union of $t_i H s_{i+1}$ and those $(H \cup G^*)$-bridges of G whose attachments are all contained in $V(t_i H s_{i+1})$. See Figure 4. Because G is $(4, H)$-connected, $I_i = t_i H s_{i+1}$ if $|V(t_i H s_{i+1})| \leq 3$. For each $1 \leq j \leq m$, let U_j denote the union of $s_j H t_j$, D_j, and those $(H \cup G^*)$-bridges of G whose attachments are all contained in $V(s_j H t_j) \cup W_j$.

Note that $|V(U_j \cap T^*)| = |V(D_j \cap T^*)| \leq 2$. Because $e^* \in E(T^*)$, we see that when $W_j \not\subseteq V(T^*)$ then $e \notin E(s_j H t_j)$. Also note that $I_i - \{t_i, s_{i+1}\}$ $(1 \leq i \leq m-1)$ and $U_j - (V(U_j \cap T^*) \cup \{s_j, t_j\})$ $(1 \leq j \leq m)$ are pairwise disjoint.

We shall construct the desired path P_k by finding the following paths: a path R_i in I_i from t_i to s_{i+1} (for each $1 \leq i \leq m-1$), a path Q_j in $U_j - V(T^* \cap U_j)$ from s_j to t_j (for each $2 \leq j \leq m-1$), a path Q_1 in U_1 from x_1 to t_1 through $s_1 = x$, and a path Q_m in U_m from s_m to y_k through $t_m = y$.

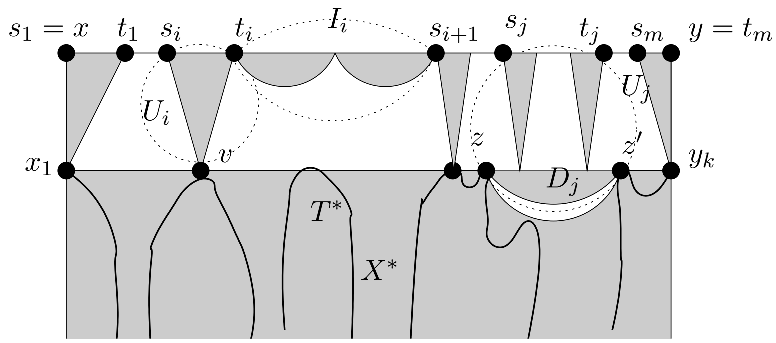


Figure 4: The graphs X^*, T^*, I_i and U_j.

Claim 1. For each $1 \leq i \leq m-1$, there is a t_iHs_{i+1}-Tutte path R_i in I_i from t_i to s_{i+1} such that $e \in E(R_i)$ whenever $e \in E(t_iHs_{i+1})$.

If $|V(t_iHs_{i+1})| \leq 3$ then $I_i = t_iHs_{i+1}$; and hence $R_i := t_iHs_{i+1}$ is the desired path. Now assume $|V(t_iHs_{i+1})| \geq 4$. Note that $I_i \cap H^* = \emptyset$, and every cut vertex of I_i must separate t_i from s_{i+1} (since G is 2-connected). Hence, $I_i^* := I_i + t_is_{i+1}$ is 2-connected. We may view I_i^* as a plane graph in which $C_i := t_iHs_{i+1} + t_is_{i+1}$ is a facial cycle. By Lemma 3.1, there is a C_i-Tutte path R_i in I_i^* from t_i to s_{i+1} and through an edge of t_iHs_{i+1} (chosen to be e if $e \in E(t_iHs_{i+1})$). It is easy to see that $t_is_{i+1} \notin E(R_i)$, and so, R_i is the desired path for Claim 1.

Claim 2. For each $2 \leq j \leq m-1$, $U_j - V(U_j \cap T^*)$ contains an s_j-t_j path Q_j such that $Q_j \cup (U_j \cap T^*)$ is an s_jHt_j-Tutte subgraph of U_j, and $e \in E(Q_j)$ whenever $e \in E(s_jHt_j)$.

If $s_j = t_j$, then let $Q_j := s_jHt_j$. In this case, $|V(U_j \cap H)| = 1$ and $e \notin E(s_jHt_j)$. Hence, since $|V(U_j \cap T^*)| \leq 2$, Q_j is the desired path for Claim 2. So we may assume $s_j \neq t_j$. We have two cases to consider.

First, assume $W_j \subseteq V(T^*)$. Then $|W_j| = 1$. Let v be the only vertex in W_j. Since G is 2-connected, any cut vertex of U_j must separate v from s_jHt_j. Hence, $U_j^* := U_j + s_jv$ is 2-connected; and we may assume that U_j^* is a plane graph in which s_jHt_j and s_jv are contained in a facial cycle C_j. (See U_i, s_i, t_i in Figure 4.) By Lemma 3.2 there is an s_jHt_j-Tutte path Q_j^* in U_j^* from t_j to v such that $s_jv \in E(Q_j^*)$, and $e \in E(Q_j^*)$ when $e \in E(s_jHt_j)$. Let $Q_j := Q_j^* - v$. Then $Q_j \subseteq U_j - V(U_j \cap T^*)$. It is easy to check that Q_j is the desired path for Claim 2.

Now assume $W_j \not\subseteq V(T^*)$. Then $e \notin E(s_jHt_j)$ (see the paragraph following the definition of U_j), and $W_j \subseteq V(D_j) - V(T^* \cap D_j)$ where D_j is a T^*-bridge of X^* containing an edge of H^*. Since T^* is an H^*-Tutte subgraph of X^*, D_j has exactly two attachments on T^*, say z and z'. Without loss of generality, we may assume that x_1, z, z', y_k occur on H^* in order. (See Figure 4.) Since G is 2-connected, any cut vertex of U_j either separates s_jHt_j from D_j or separates z from z'. Hence, $U_j^* := U_j + \{s_jz, z't_j\}$ is 2-connected; and we may assume that U_j^* is a plane graph in which $s_j, s_jHt_j, t_j, t_jz', z', z, zs_j$ occur on a facial cycle C_j in clockwise order. By Lemma 3.2, there is a zC_jz'-Tutte path Q_j^* in U_j^* from z' to z such that $\{s_jz, z't_j\} \subseteq E(Q_j^*)$. Let $Q_j := Q_j^* - \{z, z'\}$. Then $Q_j \subseteq U_j - V(T^* \cap U_j)$ and Q_j is a path from s_j to t_j. It is easy to check that Q_j is the desired path for Claim 2.

Claim 3. There is an s_1Ht_1-Tutte path Q_1 in U_1 from x_1 to t_1 such that $x \in V(Q_1)$, and $e \in E(Q_1)$ when $e \in E(s_1Ht_1)$.

If $|V(U_1)| = 2$, then $s_1 = t_1 = x$ and $e \notin E(U_1)$; in this case, $Q_1 := U_1$ is the desired path for Claim 3. Now assume that $|V(U_1)| \geq 3$. Then $U_1^* := U_1 + t_1x_1$ is 2-connected; and we may assume that U_1^* is a plane graph in which $C_1 := (x_1Fs_1 \cup s_1Ht_1) + t_1x_1$ is a facial cycle with $x_1C_1t_1 = x_1Fs_1 \cup s_1Ht_1$. (Recall that F is the frame of G.) By Lemma 3.2, there is an $x_1C_1t_1$-Tutte path Q_1 in U_1^* from x_1 to t_1 such that $x \in V(Q_1)$, and $e \in E(Q_1)$ when $e \in E(s_1Ht_1)$. It is easy to see that $t_1x_1 \notin E(Q_1)$, and so, Q_1 is the desired path for Claim 3.

Note that $t_m = y$ and $s_1 = x$. By applying the same argument as for Claim 3, with U_m, s_m, y, y_k playing the roles of U_1, t_1, x, x_1, respectively, we can prove

Claim 4. There is an s_mHt_m-Tutte path Q_m in U_m from y_k to s_m such that $y \in V(Q_m)$, and $e \in E(Q_m)$ when $e \in E(s_mHt_m)$.

Let $P_k := (\bigcup_{i=1}^{m-1} R_i) \cup (\bigcup_{j=1}^{m} Q_j)$ and let $T := T^* \cup P_k$. Then $T = T' \cup (\bigcup_{i=1}^{k} P_i)$. It is straightforward to check that any T-bridge D of $G - (V(B) - V(X))$ is one of the following: a subgraph induced by an edge of $G - E(X)$ with both incident vertices in X; or a T^*-bridge of X^* with $(V(D) - V(T^*)) \cap W = \emptyset$; or an R_i-bridge of I_i for some $1 \leq i \leq m-1$; or a $(Q_j \cup (U_j \cap T^*))$-bridge of U_j for some $1 \leq j \leq m$. Hence, it is easy to see that D has at most three attachments on T, and if D contains an edge of H then D has just two attachments on T. Therefore, T is an H-Tutte subgraph of $G - (V(B) - V(X))$. From the above claims, $e \in E(T)$ and $\{x, y\} \subseteq V(T)$.

Clearly, $T' \subseteq T$; so (i) of (4) holds. Now let $z \in V(T) - V(T')$. Then $z \in V(R_i)$ for some $1 \leq i \leq m-1$, or $z \in V(P_l)$ for some $1 \leq l \leq k-1$, or $z \in V(Q_j)$ for some $1 \leq j \leq m$. Therefore, either $z \notin V(X)$, or z is contained in $U_j \cap B_i$ for some $1 \leq j \leq m$ and $1 \leq i \leq k$. In the latter case, $z \in V(Z)$ for some T'-bridge Z of X containing an edge of H'. So (ii) of (4) holds. Since $T - (V(T') - \{x_i, y_i : 1 \leq i \leq k\}) = \bigcup_{i=1}^{k} P_i$, (iii) of (4) holds. ∎

The statement and proof of the next lemma are similar to (but more complicated than) those of Lemma 4.1. It says that given a 4-tuple associated with a graph G one can construct 4-tuples associated with graphs $B_1, \ldots, B_k$ (which are almost subgraphs of G) such that a certain Tutte subgraph T' of $\bigcup_{i=1}^{k} B_i$ can be extended to a Tutte subgraph T of G. When applied later, the closure of T' will be a disjoint union of circles, and the closure of T will be an arc. However this lemma is stated so that we can also apply it to subgraphs of G (see condition (4)). Note the notation B_i', H_i', x_i', y_i' in the statement; it is selected (partly) to avoid confusion when Lemmas 4.1 and 4.2 are applied together. See Figure 5 for an illustration.

Lemma 4.2 *Let (G, H, x, y) be a 4-tuple such that G is $(4, H)$-connected, and let $e \in E(H)$. Then there exist 4-tuples (B_i', H_i', x_i', y_i') $(1 \leq i \leq k)$ and there exist paths $u_iv_iw_i$ on H_i' such that*

(1) *B_i' is $(4, H_i')$-connected, $V(H_i' \cap H) = \{v_i\}$, and B_i' is an induced subgraph of $(G - V(H - v_i)) + \{v_iu_i, v_iw_i\}$;*

(2) *no edge of G joins a vertex of $B_i' - V(H_i')$ to a vertex of $G - V(B_i')$;*

(3) *$G - V(B')$ is finite, where $B' := \bigcup_{i=1}^{k} B_i'$, and $B_i' \cap B_l' = \emptyset$ for all $1 \leq i < l \leq k$ unless $l = i+1$, $v_i = v_{i+1}$, and $V(B_i' \cap B_{i+1}') = \{v_i = v_{i+1}\}$;*

(4) *for any subgraph (finite or infinite) X of B' containing $H' := \bigcup_{i=1}^{k} H_i'$ and for any H'-Tutte subgraph (finite or infinite) T' of X containing $\{v_iu_i, x_i', y_i' : 1 \leq i \leq k\}$ such that each v_i has exactly one neighbor in $(T' \cap B_i') - u_i$, there exists an H-Tutte subgraph T of $G - (V(B') - V(X))$ containing e, such that*

(i) $T' - \{v_i : 1 \le i \le k\} \subseteq T$,

(ii) *for any* $z \in V(T) - V(T')$, *either* $z \notin V(X)$, *or* $z \in V(Z)$ *for some* T'*-bridge* Z *of* X *containing an edge of* H', *and*

(iii) *if* w'_i, u'_i *denote the neighbors of* v_i *in* $T' \cap B'_i$ *(hence* $u_i \in \{u'_i, w'_i\}$*) such that* x'_i, w'_i, u'_i, y'_i *occur on* H'_i *in order, then* $T - V(T' - \{u'_i, v_i, w'_i : 1 \le i \le k\})$ *is the disjoint union of* u'_{i-1}*-*w'_i *paths (one for each* $1 \le i \le k+1$*), where* $u'_0 = x$ *and* $w'_{k+1} = y$.

Proof. Let F be the frame of (G, H, x, y), and let F_x and F_y denote the H-bridges of F containing x and y, respectively. Since G is 2-connected, there exists a path H^* in $G - V(H)$ from some $x' \in V(F_x - x)$ to some $y' \in V(F_y - y)$. Note that $C := H \cup yFy' \cup H^* \cup x'Fx$ is a cycle in G. We select H^*, x' and y' such that $I_G(C)$ is minimal. Let $G^* := G - V(I_G(C) - V(H^*))$. See Figure 2.

By the minimality of $I_G(C)$, any $(H \cup G^*)$-bridge of G has at most one attachment on G^* which, by planarity, must be on H^*. Therefore, because G is $(4, H)$-connected, any $(H \cup G^*)$-bridge of G has at least one attachment on H, and if an $(H \cup G^*)$-bridge of G has only one attachment on H then it is induced by a single edge.

Because G is 2-connected and by planarity, every cut vertex of G^* is contained in $H^* - \{x', y'\}$. Let $B_1, \dots, B_k$ be the infinite blocks of G^* and let x_i, y_i be distinct vertices of B_i such that $x_1 = x'$ and $y_k = y'$, $x_1, y_1, \dots, x_k, y_k$ occur on H^* in order, and $x_2, \dots, x_k, y_1, \dots, y_{k-1}$ are cutvertices of G^*. Since G is $(4, H)$-connected and by planarity, G^* is $(4, H^*)$-connected. Hence for each $1 \le i \le k-1$ with $y_i \neq x_{i+1}$, there is a unique finite $\{y_i, x_{i+1}\}$-bridge of G^*, denoted by Y_i.

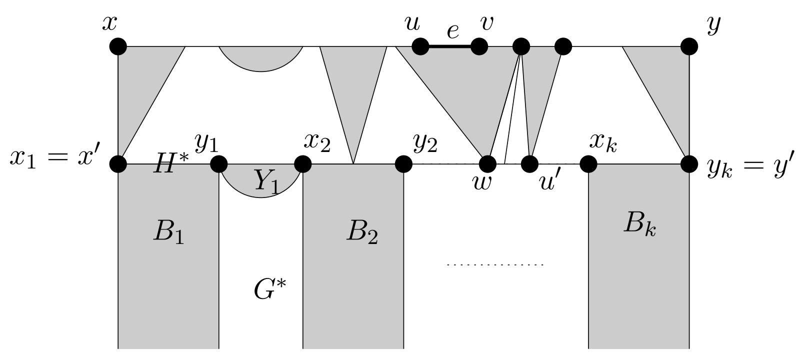


Figure 5: G^* and the vertices w and u'.

Let u, v be the vertices of G incident with e, and we may assume that x, u, v, y occur on H in order. Since G is $(4, H)$-connected, there exist two disjoint paths from xHu to H^* or two disjoint paths from vHy to H^*, both internally disjoint from $H \cup G^*$. By symmetry, we may assume that

$(*)$ *there exist two disjoint paths in* G *from* vHy *to* $w, u' \in V(H^*)$ *and internally disjoint from* $H \cup G^*$*; and we choose these paths so that* $x'H^*w$ *is minimal and, subject to this,* wH^*u' *is minimal.*

Then by planarity, x', w, u', y' occur on H^* in order. See Figure 5. By minimality of $x'H^*w$ and by planarity, any path from $x'H^*w - w$ to H internally disjoint from $H \cup G^*$ must intersect xHu. Because G is $(4, H)$-connected and by minimality of wH^*u', all $(H \cup G^*)$-bridges of G with an attachment in $wH^*u' - \{w, u'\}$ must be induced by single edges that are incident with a common vertex of H.

Note that either there is some $1 \le t \le k$ such that $w \in V(x_tH^*y_t - y_t)$, or there exists some $1 \le t < k$ such that $w \in V(y_tH^*x_{t+1} - x_{t+1})$. In either case, since G is $(4, H)$-connected, $u' \in V(wH^*y_{t+1} - y_{t+1})$.

Before we define the 4-tuples (B_i', H_i', x_i', y_i') from B_i, we find the special paths in (a) and (b) below according to the location of w. (We will use (a) in Steps 3, 4 and 5, and will use (b) in Steps 1, 2, and 6.) Let W denote the set of attachments on H^* of $(H \cup G^*)$-bridges of G.

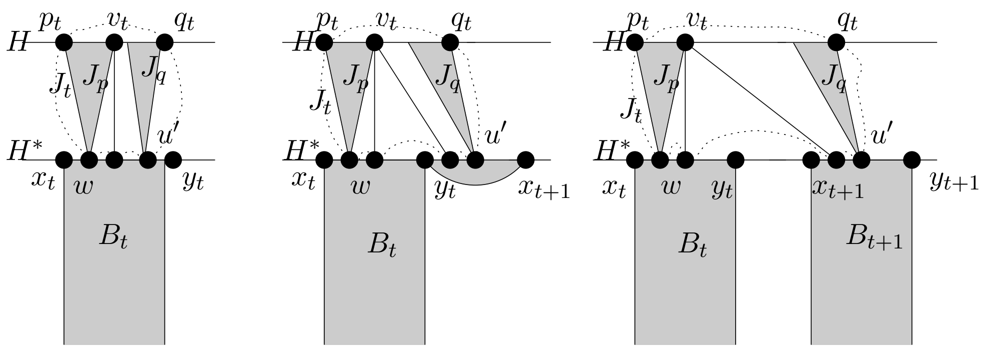


Figure 6: J_t, p_t, and q_t when $w \in V(x_tH^*y_t - y_t)$.

Suppose $w \in V(x_tH^*y_t - y_t)$. Figure 6 illustrates the three cases according to the location of u': $u' \in V(wH^*y_t)$; $u' \in V(y_tH^*x_{t+1}) - \{y_t, x_{t+1}\}$; $u' \in V(x_{t+1}H^*y_{t+1} - y_{t+1})$. Note that when $u' \in V(x_{t+1}H^*y_{t+1} - y_{t+1})$, it follows from $(4, H)$-connectedness of G that either $y_t = x_{t+1}$, or Y_t is induced by the edge y_tx_{t+1}. Define $p_t, q_t \in V(H)$ with p_tHq_t maximal such that $\{p_t, w\}$ and $\{q_t, u'\}$ each are contained in an $(H \cup G^*)$-bridge of G. Note that $p_t \ne q_t$ by the choice of w and u'. By planarity, x, p_t, q_t, y occur on H in order. Let J_t denote the union of p_tHq_t and those $(H \cup G^*)$-bridges of G whose attachments are all contained in $p_tHq_t \cup wH^*u'$. See Figure 6, where J_t is in the disc bounded by the dotted closed curve. Let $v_t \in V(H)$ with v_tHy minimal such that $\{v_t, w\}$ is contained in an $(H \cup G^*)$-bridge of G. Note that by $(*)$ all $(H \cup G^*)$-bridges of G with an attachment $w^* \in V(wH^*u') - \{w, u'\}$ must be induced by the edge v_tw^*.

(a) *If $w \in V(x_tH^*y_t - y_t)$ then for any $w_t' \in W \cap V(wH^*u' - u')$, there exist disjoint paths P_t, Q_t in J_t from p_t, q_t to w_t', u', respectively, such that $V(P_t \cup Q_t) \cap W = \{w_t', u'\}$, $e \in E(P_t \cup Q_t)$ when $e \in E(J_t)$, $v_t \in V(P_t \cup Q_t)$, and $(P_t \cup Q_t) + w$ is a p_tHq_t-Tutte subgraph of J_t.*

To prove (a), let J_p and J_q denote the v_t-bridges of J_t containing $\{p_t, w\}$ and $\{q_t, u'\}$, respectively. By the choice of w, u', p_t and q_t, $J_p - q_t$ has a path from w to p_t and through e when $e \in E(J_t)$. So let J' denote the block of $(J_p - q_t) + wp_t$ containing a cycle through wp_t, and let $v' \in V(J' \cap H)$ with $v'Hq_t$ minimal. We may assume that p_t, p_tHv', v', w occur on its outer cycle in clockwise order. Then $e \in E(J')$ when $e \in E(J_t)$. (Note that $v' = v_t$ if, and only if, $q_t \notin V(J_p)$.) First, suppose $w_t' = w$. In J', we apply Lemma 3.2 to find a p_tHv'-Tutte path P_t from w to p_t through v' and also e when $e \in E(J')$. If $q_t = v_t$, then let Q_t be the path induced by the edge $u'q_t$; and otherwise, in $J_q + u'v_t$, we use Lemma 3.1 to find a v_tHq_t-Tutte path Q_t' from q_t to v_t through $u'v_t$, and let $Q_t := Q_t' - v_t$. Then P_t and Q_t are the desired paths for (a). Now suppose $w_t' \ne w$. Then, since $u' \ne w_t'$, we have $q_t \ne v_t$ by the choice of u'; and hence, $J' = J_p + wp_t$. In J' we use Lemma 3.2 to find a p_tHv_t-Tutte path P_t' from w to v_t such that $wp_t \in E(P_t')$, and $e \in E(P_t')$

when $e \in E(J_t)$; and let $P_t := (P'_t - w) + \{w'_t, w'_t v_t\}$. In $J_q + u'v_t$, we use Lemma 3.1 to find a $v_t H q_t$-Tutte path Q'_t from q_t to v_t through $v_t u'$; and let $Q_t := Q'_t - v_t$. Then P_t and Q_t are the desired paths for (a).

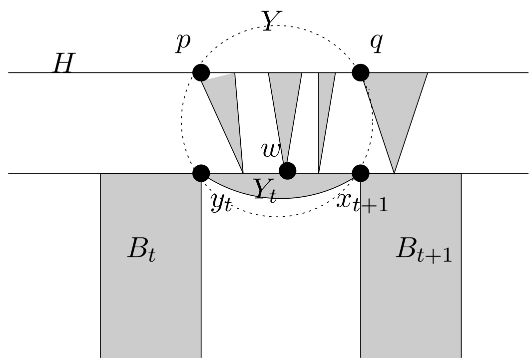


Figure 7: Y, p, q when $w \in V(y_t H^* x_{t+1} - x_{t+1})$.

Suppose $w \in V(y_t H^* x_{t+1} - x_{t+1})$. See Figure 7. Let $p, q \in V(H)$ such that there exist paths from p, q to $y_t H^* x_{t+1} - x_{t+1}, x_{t+1} H^* y_{t+1} - x_{t+1}$, respectively, and internally disjoint from $H \cup G^*$, and subject to this, xHp and xHq are minimal. By planarity, x, p, q, y occur on H in order. Let Y denote the union of pHq, Y_t, and those $(H \cup G^*)$-bridges of G whose attachments are all contained in $V(pHq \cup y_t H^* x_{t+1})$. In Figure 7, Y is contained in the disk bounded by the dotted closed curve.

(b) *If $w \in V(y_t H^* x_{t+1} - x_{t+1})$ then there is a path Q in $(Y + qx_{t+1}) - y_t$ from p to q such that $e \in E(Q)$ whenever $e \in E(pHq)$, and $Q + y_t$ is a pHq-Tutte subgraph of $Y + qx_{t+1}$. Note that one of the following holds: $x_{t+1}q \in E(Q)$; or $x_{t+1}q \notin E(Q)$ but $x_{t+1} \in V(Q)$; or $x_{t+1} \notin V(Q)$.*

To see (b) we note that, since G is 2-connected, any cut vertex of Y either separates pHq from Y_t or separates y_t from x_{t+1}. Hence, $Y^* := Y + \{py_t, qx_{t+1}\}$ is 2-connected; and we may assume that Y^* is a 2-connected plane graph in which $pHq + \{x_{t+1}, y_t, qx_{t+1}, py_t\}$ is contained in a facial cycle C^* of Y^* and y_t, p, pHq, q, x_{t+1} occur on C^* in clockwise order. By Lemma 3.2, there is a $y_t C^* q$-Tutte path Q^* in Y^* from q to y_t such that $y_t p \in E(Q^*)$, and $e \in E(Q^*)$ when $e \in E(pHq)$. Then $Q := Q^* - y_t$ is the desired path for (b).

In (c) and (d) below, we define the path L_t in Y_t by considering the locations of w and u'.

(c) *If $w \in V(x_t H^* y_t - y_t)$ and $u' \in V(y_t H^* x_{t+1} - y_t)$, then there is a path L_t in Y_t from u' to y_t such that $L_t + x_{t+1}$ is a $y_t H^* x_{t+1}$-Tutte subgraph of Y_t, and $x_{t+1} \notin V(L_t)$ when $u' \neq x_{t+1}$.*

If $u' = x_{t+1}$ then $y_t = x_{t+1}$ or $y_t x_{t+1}$ is an edge (since G is $(4, H)$-connected); and $L_t := Y_t$ is the desired path. So assume $u' \neq x_{t+1}$. We may view $Y_t + y_t x_{t+1}$ as a 2-connected plane graph in which $y_t H^* x_{t+1} + y_t x_{t+1}$ is a facial cycle. See the middle graph of Figure 6. By Lemma 3.1, there is a $y_t H^* x_{t+1}$-Tutte path L'_t in $Y_t + y_t x_{t+1}$ from u' to x_{t+1} through $y_t x_{t+1}$. Now $L_t := L'_t - x_{t+1}$ is the desired path for (c).

(d) *For all situations other than those described in (c), we define $L_t = \emptyset$.*

We now define $B'_i, H'_i, u_i, v_i, w_i, x'_i, y'_i$ for $1 \le i \le k$, which is done in six steps. At each step, we show that (1) and (2) hold. When $B'_i, H'_i, u_i, v_i, w_i, x'_i, y'_i$ are defined for all $1 \le i \le k$, we shall show that (3) and (4) also hold. Step 1 and Step 2 take care of all $i \notin \{t, t+1\}$, as well as some cases when $i \in \{t, t+1\}$. Steps 3, 4, and 5 deal with the remaining cases for $i = t$; and Step 6 deals with the remaining case for $i = t+1$. Note that when $w \in V(x_t H^* y_t - y_t)$ and $u' \in V(x_{t+1} H^* y_{t+1} - \{x_{t+1}, y_{t+1}\}$, B_t and B_{t+1} are contained in B'_t, and so B'_{t+1} need not be defined; this is done in step 4.

Step 1. *Definition of* $B'_i, H'_i, u_i, v_i, w_i, x'_i, y'_i$ *if*

(i) $1 \le i \le t-1$ *and* $y_i \neq x_{i+1}$, *or*

(ii) $t+2 \le i \le k$ *and* $x_i \neq y_{i-1}$, *or*

(iii) $i = t$ *and* $w \in V(y_t H^* x_{t+1} - x_{t+1})$, *or*

(iv) $i = t+1$, $w \in V(x_t H^* y_t - y_t)$, *and* $u' \in V(w H^* x_{t+1} - x_{t+1})$, *or*

(v) $i = t+1$, $w \in V(y_t H^* x_{t+1} - x_{t+1})$, *and* $x_{t+1} \notin V(Q)$ *(where* Q *is defined in (b)).*

Choose w_i, u_i from $V(x_i H^* y_i - \{x_i, y_i\})$ with $w_i H^* u_i$ minimal such that there are disjoint paths from w_i, u_i to H and internally disjoint from $H \cup G^*$. Such w_i and u_i exist because G is $(4, H)$-connected. We may assume that x_i, w_i, u_i, y_i occur on H^* in order. By the minimality of $w_i H^* u_i$, no $(H \cup G^*)$-bridge of G has an attachment in $w_i H^* u_i - \{w_i, u_i\}$. Let $v_i \in V(H)$ with $v_i H y$ minimal such that $\{v_i, w_i\}$ is contained in an $(H \cup G^*)$-bridge of G. See Figure 8.

Let $x'_i := x_i$ and $y'_i := y_i$, let $B'_i := B_i + \{v_i, v_i u_i, v_i w_i\}$, and $H'_i := (x'_i H^* w_i \cup u_i H^* y_i) + \{v_i, v_i u_i, v_i w_i\}$. Clearly, (B'_i, H'_i, x'_i, y'_i) is a 4-tuple. Since no $(H \cup G^*)$-bridge of G has an attachment in $w_i H^* u_i - \{w_i, u_i\}$, B'_i is an induced subgraph of $(G - V(H - v_i)) + \{v_i u_i, v_i w_i\}$. Because G is $(4, H)$-connected, B'_i is $(4, H'_i)$-connected. So B'_i and H'_i satisfy (1). By planarity, B'_i and H'_i also satisfy (2). Moreover

(1a) *for any* H'_i*-Tutte subgraph* T'_i *of* B'_i *containing* $\{v_i u_i, x'_i, y'_i\}$, *we must have* $w_i \in V(T'_i)$.

For, otherwise, the T'_i-bridge of B'_i containing w_i would have three attachments on T'_i, namely v_i and two on H^* (since $x'_i, y'_i \in V(T'_i)$).

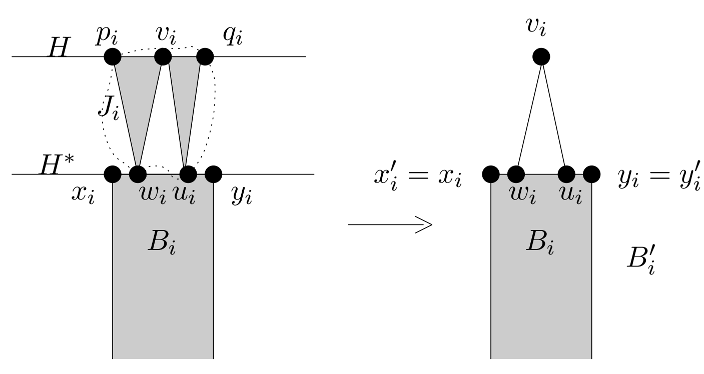


Figure 8: $B'_i, H'_i, u_i, v_i, w_i$ and x'_i in Step 1.

Let $p_i, q_i \in V(H)$ with $p_i H q_i$ maximal such that $\{p_i, w_i\}$ and $\{q_i, u_i\}$ each are contained in an $(H \cup G^*)$-bridge of G. See Figure 8. By the choice of w_i and u_i, $p_i \neq q_i$. By planarity, x, p_i, q_i, y

occur on H in order. By the choice of w and u' and by the restriction on i, $e \notin E(p_iHq_i)$. Let J_i denote the union of p_iHq_i and those $(H \cup G^*)$-bridges of G whose attachments are all contained in $V(p_iHq_i) \cup \{w_i, u_i\}$. Then

(1b) *there exist disjoint paths P_i, Q_i in J_i from p_i, q_i to w_i, u_i, respectively, such that $v_i \in V(P_i \cup Q_i)$, and $P_i \cup Q_i$ is a p_iHq_i-Tutte subgraph of J_i.*

To prove (1b), we note that $J_i + w_iu_i$ is 2-connected; and we may assume that it is a plane graph in which w_iu_i and p_iHq_i are contained in a facial cycle. By applying Lemma 3.1, there is a p_iHq_i-Tutte path S_i from p_i to q_i such that $w_iu_i \in E(S_i)$. Let P_i and Q_i denote the components of $S_i - w_iu_i$ containing p_i and q_i, respectively. Then by planarity, P_i is from p_i to w_i and Q_i is from q_i to u_i. Clearly, $P_i \cup Q_i$ is a p_iHq_i-Tutte subgraph of J_i.

Note that $v_i \in V(P_i \cup Q_i)$; as otherwise the $(P_i \cup Q_i)$-bridge of $J_i + w_iu_i$ containing v_i would have at least three attachments (since $\{v_i, w_i\}$ is contained in an $(H \cup G^*)$-bridge of G). This concludes Step 1.

We need to define $B'_i, H'_i, u_i, v_i, w_i, x'_i, y'_i$ that are not defined in Step 1: $i = t$, or $i = t+1$, or $x_i = y_{i-1}$. Note that for some cases (according to the location of w, u'), B'_t or B'_{t+1} is defined in Step 1. In the remainder of this proof, when $w \in V(x_tH^*y_t - y_t)$, we define $w_t := w$; and recall that $v_t \in V(H)$ with v_tHy minimal such that $\{v_t, w_t\}$ is contained in an $(H \cup G^*)$-bridge of G (see Figure 6).

Step 2. *Definition of $B'_i, H'_i, u_i, v_i, w_i, x'_i, y'_i$ if*

- *(i)* $i \geq t+2$, $x_i = y_{i-1}$, *or*
- *(ii)* $i = t+1$, $w \in V(x_tH^*y_t - y_t)$, *and* $u' = x_{t+1}$, *or*
- *(iii)* $i = t+1$, $w \in V(y_tH^*x_{t+1} - x_{t+1})$, $x_{t+1} \in V(Q)$, *and* $qx_{t+1} \notin E(Q)$, *or*
- *(iv)* $i \leq t-1$ *and* $y_i = x_{i+1}$.

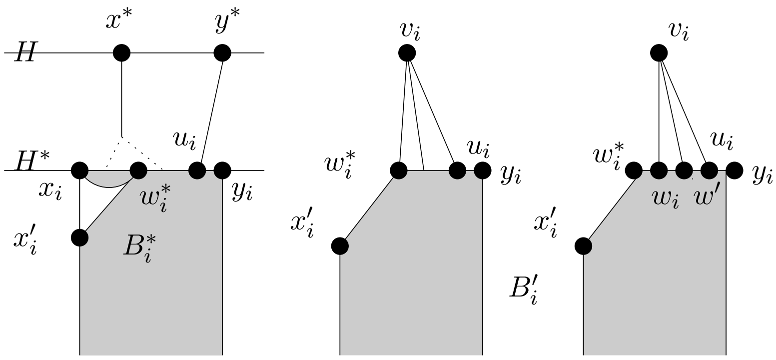
First, we define $B'_i, H'_i, u_i, v_i, w_i, x'_i, y'_i$ when (i) or (ii) or (iii) occurs; the definition when (iv) occurs is symmetric to (i) (and will be sketched).

Because (B_i, H_i, x_i, y_i) is a 4-tuple and since G^* is $(4, H^*)$-connected and has no dividing cycles, $B_i - x_i$ has a unique infinite block which we denote by B^*_i; so every $(B^*_i + x_i)$-bridge of B_i is finite. See Figure 9. Let F^* denote the double ray whose vertices and edges are all incident with the face of B^*_i that is not a face of B_i. (So F^* and the frame of (B_i, H_i, x_i, y_i) differ by a finite path.) Let $x'_i, w^*_i \in V(F^*)$ be the attachments of $(B^*_i + x_i)$-bridges of B_i such that $x'_iF^*w^*_i$ is maximal, with $x'_i \notin V(H^*)$ and $w^*_i \in V(H^*)$.

Since G is $(4, H)$-connected, there exist $u_i \in V(w^*_iH^*y_i - w^*_i)$, $w' \in V(x_iH^*u_i - \{x_i, u_i\})$, and disjoint paths in G from w', u_i to $x^*, y^* \in V(H)$, respectively, and internally disjoint from $H \cup G^*$. We choose such w' and u_i that

$w^*_iH^*u_i$ is minimal and subject to this, $w'H^*u_i$ and x^*Hy^* are minimal.

Then each $(H \cup G^*)$-bridge of G with an attachment w^* in $w'H^*u_i - \{w', u_i\}$ is induced by the edge y^*w^* (otherwise, w^*, u_i contradicts the choice of w', u_i). So by the choice of w', u_i, no $(H \cup G^*)$-bridge of G has an attachment in $(w'H^*u_i - \{w', u_i\}) \cap (w^*_iH^*u_i - \{w^*_i, u_i\})$; otherwise, w' and such an attachment contradicts the choice of w', u_i.



(a) $w_i = w_i^*$ (b) $w_i \neq w_i^*$

Figure 9: $B_i', H_i', u_i, v_i, w_i, x_i', y_i'$ in Step 2.

If $w' \in V(x_iH^*w_i^*)$ we define $w_i = w_i^*$ (see Figure 9(a)). If $w' \in V(w_i^*H^*u_i - \{w_i^*, u_i\})$ then each $(H \cup G^*)$-bridge of G with an attachment, say w^*, in $x_iH^*w' - \{x_i, w'\}$ must be induced by the edge x^*w^* (otherwise, w^*, w' contradict the choices of w', u_i). In this case, since G is $(4, H)$-connected, the $(B_i^* + x_i)$-bridge of B_i containing $x_iH^*w_i^*$ is induced by the edge $x_iw_i^*$. We let $w_i \in W \cap V(w_i^*H^*w')$ such that $w_i^*H^*w_i$ is minimal (see Figure 9(b)).

Let $y_i' = y_i$. Let $v_i \in V(H)$ with v_iHy minimal such that $\{v_i, w'\}$ is contained in some $(H \cup G^*)$-bridge of G. Note that $v_i \in \{x^*, y^*\}$. Define $B_i' := B_i^* + \{v_i, v_iw_i, v_iu_i, v_iw^* : w^* \in W \cap V(w_iH^*u_i)\}$ and $H_i' := (x_i'F^*w_i \cup u_iF^*y_i) + \{v_i, v_iw_i, v_iu_i\}$. Note that (B_i', H_i', x_i', y_i') is a 4-tuple. Clearly B_i' is an induced subgraph of $(G - V(H - v_i)) + \{v_iw_i, v_iu_i\}$. Since B_i is $(4, H_i)$-connected, B_i' is $(4, H_i')$-connected. So (1) holds for B_i' and H_i'. By planarity, (2) also holds for B_i' and H_i'.

We claim that

(2a) *for any H_i'-Tutte subgraph T_i' of B_i' containing $\{v_iu_i, x_i', y_i'\}$, we must have $\{w_i, w_i^*\} \subseteq V(T_i')$.*

For otherwise, we may assume that w_i or w_i^* is contained in a T_i'-bridge Z of B_i' with exactly two attachments which are on $x_i'F^*u_i$ (since $x_i', u_i \in V(T_i')$). If $w_i \notin V(T_i')$ then we may assume that $w_i \in V(Z)$; in this case v_i must also be an attachment of Z, a contradiction. So $w_i \in V(T_i')$. Then $w_i^* \in V(Z)$ and $w_i^* \notin V(T_i')$. Hence $w_i \in V(w_i^*H^*u_i) - \{w_i^*, u_i\}$, and no $(H \cup G^*)$-bridge of G has an attachment in $w_i^*H^*w_i - w_i$. Thus, since $x_iH^*w_i^*$ is induced by the edge $x_iw_i^*$, $V(Z \cap T_i') \cup \{x_i\}$ is a 3-cut in G, contradicting the assumption that G is $(4, H)$-connected.

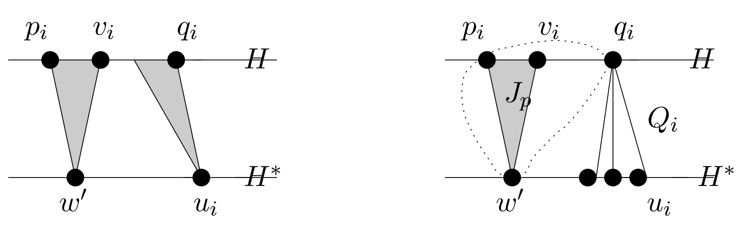


Figure 10: The graph J_i.

Let $p_i, q_i \in V(H)$ with p_iHq_i maximal such that $\{p_i, w'\}$ and $\{q_i, u_i\}$ each are contained in an $(H \cup G^*)$-bridge of G. By planarity, x, p_i, x^*, y^*, q_i, y occur on H in order. Note that $p_i \neq q_i$ by the choice of w' and u_i. (Clearly, $e \notin E(p_i H q_i)$ since $i \geq t+1$.) Let J_i denote the union of $p_i H q_i$ and those $(H \cup G^*)$-bridges of G whose attachments are all contained in $V(p_i H q_i) \cup V(w' H^* u_i)$. Then

(2b) *there exist disjoint paths P_i, Q_i in J_i from p_i, q_i to w', u_i, respectively, such that $W \cap V(P_i \cup Q_i) = \{w', u_i\}$, $v_i \in V(P_i \cup Q_i)$, and $P_i \cup Q_i$ is a $p_i H q_i$-Tutte subgraph of J_i.*

If no $(H \cup G^*)$-bridge of G has an attachment contained in $w' H^* u_i - \{w', u_i\}$ (see the left graph in Figure 10) then (2b) follows from the same argument as that for (1b). So we may assume otherwise. Then by the minimality of $w' H^* u_i$, $y^* = q_i$, and each $(H \cup G^*)$-bridge of G with an attachment in $w' H^* u_i - w'$ must be induced by an edge incident with $y^* = q_i$ (see the right graph in Figure 10). Let Q_i be induced by the edge $q_i u_i$. Let J_p denote the q_i-bridge of J_i containing p_i and w'. (J_p is inside the dotted closed curve in Figure 10.) In $J_p + w' q_i$ we apply Lemma 3.1 to find a $p_i H q_i$-Tutte path P_i' from p_i to q_i through $w' q_i$; and let $P_i := P_i' - q_i$. It is easy to check that P_i, Q_i are the desired paths for (2b). In particular, $v_i \in V(P_i \cup Q_i)$; since otherwise v_i is contained in a $(P_i \cup Q_i)$-bridge of J_i with two attachments, contradicting the fact that $\{v_i, w'\}$ is contained in an $(H \cup G^*)$-bridge of G.

Let X_i denote the $(B_i^* + x_i)$-bridge of B_i containing $x_i H^* w_i^*$.

(2c) *If $w' \in V(x_i H^* w_i^*) - \{x_i, w_i^*\}$, then there exists a path N_i in $X_i - x_i$ from w' to w_i^* such that $N_i + x_i$ is an $x_i H^* w_i^*$-Tutte subgraph of X_i.*

To see this, we view $X_i + x_i w_i^*$ as a 2-connected plane graph in which $x_i H^* w_i^* + x_i w_i^*$ is a facial cycle. By Lemma 3.1, there exists an $x_i H^* w_i^*$-Tutte path N_i' in $X_i + x_i w_i^*$ from x_i to w' through $x_i w_i^*$. Then $N_i := N_i' - x_i$ is the desired path for (2c).

We now define $B_i', H_i', u_i, v_i, w_i, x_i', y_i'$ when $i \leq t-1$ and $y_i = x_{i+1}$. This is symmetric (left-right reflection) to the above case for B_{i+1}' and H_{i+1}'. So we only give a sketch. See Figure 11.

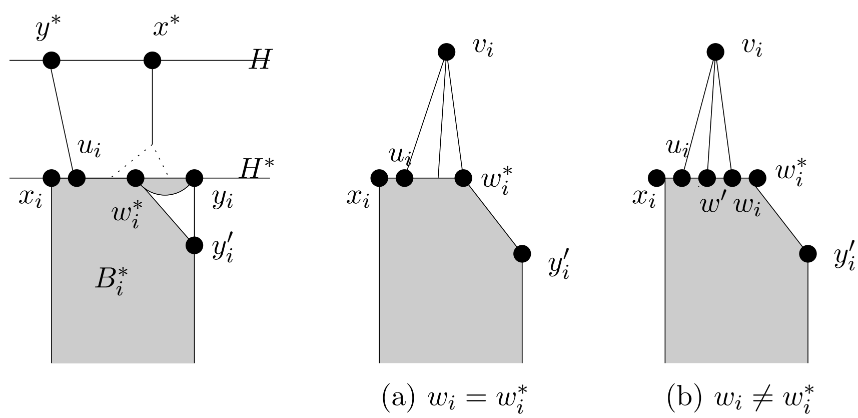


Figure 11: $B_i', H_i', u_i, v_i, w_i, x_i', y_i'$ (for $i < t$ and $y_i = x_{i+1}$) in Step 2.

Here we consider the infinite block B_i^* of $B_i - y_i$, and let F^* denote the double ray whose vertices and edges are incident with the face of B_i^* that is not a face of B_i. Let w_i^*, y_i' denote the attachments of $(B_i^* + y_i)$-bridges of B_i such that $w_i^* \in V(H^*)$, $y_i' \notin V(H^*)$, and $w_i^* F^* y_i'$ is maximal.

Since G is $(4, H)$-connected and has no dividing cycles, there exist $u_i \in V(w_i^* H^* x_i - w_i^*)$, $w' \in V(y_i H^* u_i) - \{y_i, u_i\}$, and disjoint paths in G from w', u_i to $x^*, y^* \in V(H)$, respectively, and internally disjoint from $H \cup G^*$. We choose w', u_i so that $w_i^* H^* u_i$ is minimal, and subject to this, $w' H^* u_i$ and $x^* H y^*$ are minimal.

If $w' \in V(y_i H^* w_i^*)$ then let $w_i = w_i^*$ (see Figure 11(a)). If $w' \in V(w_i^* H^* u_i) - \{w_i^*, u_i\}$ then let $w_i \in W \cap V(w_i^* H^* w')$ so that $w_i H^* w_i^*$ is minimal (see Figure 11(b)). Let $v_i \in V(H)$ with $v_i H x$ minimal such that $\{v_i, w'\}$ is contained in an $(H \cup G^*)$-bridge of G. Let $x_i' = x_i$. Define $B_i' := B_i^* + \{v_i, v_i u_i, v_i w_i, v_i w^* : w^* \in W \cap V(w_i H^* u_i)\}$ and $H_i' := (x_i' F^* u_i \cup w_i F^* y_i') + \{v_i, v_i u_i, v_i w_i\}$.

Then (B_i', H_i', x_i', y_i') is a 4-tuple. Clearly, B_i' is an induced subgraph of $(G - V(H - v_i)) + \{v_i u_i, v_i w_i\}$, and is $(4, H_i')$-connected. So (1) holds for B_i' and H_i'. By planarity, (2) also holds for B_i' and H_i'. Similar to (2a), we have

(2a′) for any H_i'-Tutte subgraph T_i' of B_i' containing $\{v_i u_i, x_i', y_i'\}$, we have $\{w_i, w_i^*\} \subseteq V(T_i')$.

Let $p_i, q_i \in V(H)$ with $p_i H q_i$ maximal such that $\{p_i, w'\}$ and $\{q_i, u_i\}$ each are contained in an $(H \cup G^*)$-bridge of G. By the choices of w' and u_i, $p_i \neq q_i$. Since $i \leq t-1$, $e \notin E(p_i H q_i)$. Let J_i denote the union of $p_i H q_i$ and those $(H \cup G^*)$-bridges of G whose attachments are all contained in $V(p_i H q_i) \cup V(w' H^* u_i)$. Then, similar to (2b), we have

(2b′) there exist disjoint paths P_i, Q_i from p_i, q_i to w', u_i, respectively, such that $W \cap V(P_i \cup Q_i) = \{w', u_i\}$, $v_i \in V(P_i \cup Q_i)$, and $P_i \cup Q_i$ is a $p_i H q_i$-Tutte subgraph of J_i.

Let X_i denote the $(B_i^* + y_i)$-bridge of B_i containing $y_i H^* w_i^*$. Similar to (2c), we have

(2c′) If $w' \in V(y_i H^* w_i^*) - \{y_i, w_i^*\}$, then there exists a path N_i in $X_i - y_i$ from w' to w_i^* such that $N_i + y_i$ is a $y_i H^* w_i^*$-Tutte subgraph of X_i.

Step 3. *Definition of $B_t', H_t', u_t, v_t, w_t, x_t', y_t'$ when $w \in V(x_t H^* y_t - y_t)$ and $u' \in V(w H^* x_{t+1} - x_{t+1})$.*

Recall $w_t = w$ and the definition of v_t preceding (a). Let $u_t := u'$ if $u' \in V(w_t H^* y_t)$ (see Figure 12(a)), and let $u_t := y_t$ if $u' \in V(y_t H^* x_{t+1}) - \{y_t, x_{t+1}\}$ (see Figure 12(b)). Define $x_t' := x_t$ and $y_t' = y_t$. Let $B_t' := B_t + \{v_t, v_t u_t, v_t w_t, v_t w^* : w^* \in W \cap V(w_t H^* u_t)\}$ and $H_t' := (x_t H^* w_t \cup u_t H^* y_t) + \{v_t, v_t w_t, v_t u_t\}$.

Clearly, (B_t', H_t', x_t', y_t') is a 4-tuple. Recall that every $(H \cup G^*)$-bridge of G with an attachment in $w_t H^* u_t - \{w_t, u_t\}$ is induced by a single edge incident with v_t. So B_t' is an induced subgraph of $(G - V(H - v_t)) + \{v_t w_t, v_t u_t\}$. Using planarity and $(4, H)$-connectedness of G, it is not hard to show that B_t' is $(4, H_t')$-connected. So B_t' and H_t' satisfy (1). By planarity, B_t' and H_t' satisfy (2) as well. By the same argument as that for (1a), we have

(3a) *every H_t'-Tutte subgraph of B_t' containing $\{v_t u_t, x_t', y_t'\}$ must contain w_t.*

Note that when we later extend an H_t'-Tutte subgraph of B_t' we will use the paths P_t, Q_t in (a), as well as the path L_t in (c).

Step 4. *Definition of $B_t', H_t', u_t, v_t, w_t, x_t', y_t'$ when $w \in V(x_t H^* y_t - y_t)$ and $u' \in V(x_{t+1} H^* y_{t+1} - \{x_{t+1}, y_{t+1}\})$. (In this case, $B_t \cup B_{t+1} \subseteq B_t'$; so B_{t+1}' need not be defined.)*

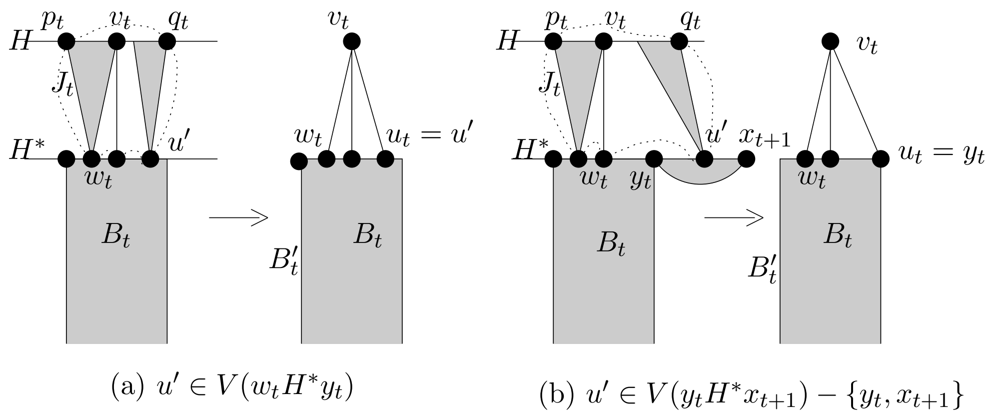


Figure 12: B'_t, w_t, and u_t in Step 3.

Suppose $w \in V(x_tH^*y_t - y_t)$ and $u' \in V(x_{t+1}H^*y_{t+1} - \{x_{t+1}, y_{t+1}\})$. Since G is $(4, H)$-connected, either $y_t = x_{t+1}$ or Y_t is induced by the edge y_tx_{t+1}. Let $w_t = w$ and recall the definition of v_t preceding (a).

Let $u_t := u'$, $x'_t := x_t$, $y'_t := y_{t+1}$, $B'_t := (B_t \cup B_{t+1}) + \{v_t, v_tu_t, v_tw_t, v_tw^* : w^* \in V(w_tH^*u_t) \cap W\}$, and $H'_t := (x'_tH^*w_t \cup u_tH^*y'_t) + \{v_t, v_tw_t, v_tu_t\}$. See Figure 13. Since G is $(4, H)$-connected, B'_t is $(4, H'_t)$-connected. Clearly, B'_t is an induced subgraph of $(G - V(H - v_t)) + \{v_tw_t, v_tu_t\}$. So B'_t and H'_t satisfy (1). By planarity, B'_t and H'_t satisfy (2). By the same argument as that for (1a), we have

(4a) *every H'_t-Tutte subgraph of B'_t containing $\{v_tu_t, x'_t, y'_t\}$ also contains w_t.*

Note that when we later extend an H'_t-Tutte subgraph of B'_t we will use the paths P_t, Q_t in (a).

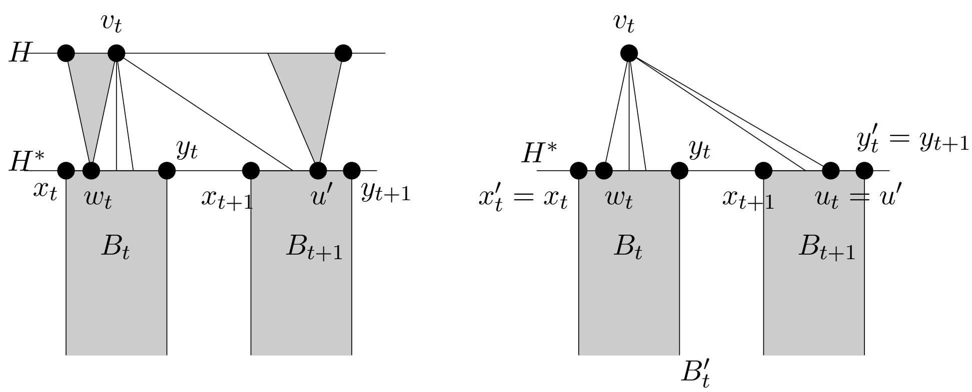


Figure 13: B'_t in Step 4.

Note that when Step 4 applies, B'_{t+1} is not defined. Once all B'_i are defined, we can simply perform a relabeling for $i \geq t + 2$ (by relabeling B'_i and H'_i with B'_{i-1} and H'_{i-1}, respectively).

Step 5. *Definition of $B'_t, H'_t, u_t, v_t, w_t, x'_t, y'_t$ when $w \in V(x_tH^*y_t - y_t)$ and $u' = x_{t+1}$.*

See Figure 14. Let $u_t := y_t$, $x'_t := x_t$, $y'_t = y_t$, $H'_t := x'_tH^*w_t + \{v_t, u_t, v_tw_t, v_tu_t\}$, and $B'_t := B_t + \{v_t, v_tu_t, v_tw^* : w^* \in V(w_tH^*u_t) \cap W\}$. Clearly, (B'_t, H'_t, x'_t, y'_t) is a 4-tuple. Recall that every $(H \cup G^*)$-bridge of G with an attachment $w^* \in V(wH^*u') - \{w, u'\}$ is induced by the edge v_tw^* of G. So B'_t is an induced subgraph of $(G - V(H - v_t)) + \{v_tu_t, v_tw_t\}$. Since G is $(4, H)$-connected, B'_t is $(4, H'_t)$-connected. Hence B'_t, H'_t satisfy (1). By planarity, B'_t and H'_t satisfy (2). Similar to (1a), we have

(5a) *every H'_t-Tutte subgraph of B'_t containing $\{v_tu_t, x'_t, y'_t\}$ also contains w_t.*

Note that when we later extend an H'_t-Tutte subgraph of B'_t we will use the paths P_t, Q_t in (a), as well as the path L_t in (c).

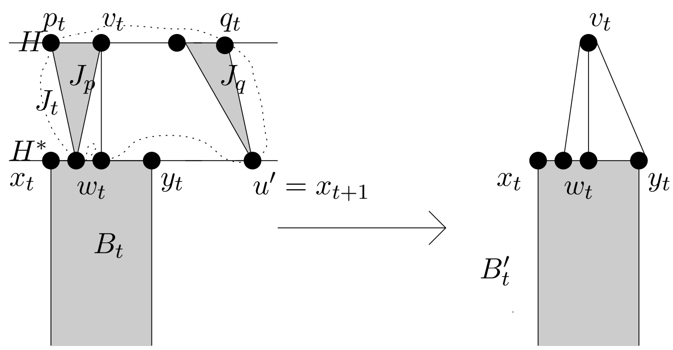


Figure 14: $w \in V(x_tH^*y_t - y_t)$ and $u' = x_{t+1}$.

Step 6. *Definition of $B'_{t+1}, H'_{t+1}, u_{t+1}, v_{t+1}, w_{t+1}, x'_{t+1}, y'_{t+1}$ when $w \in V(y_tH^*x_{t+1} - x_{t+1})$.*

Suppose $w \in V(y_tH^*x_{t+1} - x_{t+1})$. Then $y_t \neq x_{t+1}$, and we just need to define B'_{t+1} for the case $qx_{t+1} \in E(Q)$; all other cases are done in Step 1 and Step 2. Recall Figure 7, the path Q in (b), and the definition of p, q preceding (b). Let $w_{t+1} := x_{t+1}$, and let $u_{t+1} \in W \cap V(x_{t+1}H^*y_{t+1} - x_{t+1})$ such that $x_{t+1}H^*u_{t+1}$ is minimal. Note from the definition of q, $\{q, u_{t+1}\}$ is contained in an $(H \cup G^*)$-bridge of G. Let $v_{t+1} := q$, $x'_{t+1} := x_{t+1} = w_{t+1}$, and $y'_{t+1} := y_{t+1}$. Define $B'_{t+1} := B_{t+1} + \{v_{t+1}, v_{t+1}u_{t+1}, v_{t+1}w_{t+1}\}$, and let $H'_{t+1} := u_{t+1}H^*y'_{t+1} + \{w_{t+1}, v_{t+1}, v_{t+1}u_{t+1}, v_{t+1}w_{t+1}\}$. Then $(B'_{t+1}, H'_{t+1}, x'_{t+1}, y'_{t+1})$ is a 4-tuple. It is easy to check that B'_{t+1} and H'_{t+1} satisfy (1) and (2). Also, since $x'_{t+1} = w_{t+1}$,

(6a) *any H'_{t+1}-Tutte subgraph of B'_{t+1} containing $\{v_{t+1}u_{t+1}, x'_{t+1}, y'_{t+1}\}$ contains w_{t+1}.*

Let $s \in V(H)$ with sHy minimal such that $\{s, u_{t+1}\}$ is contained in an $(H \cup G^*)$-bridge of G (possibly $s = q$). Let J_{t+1} denote the union of qHs and those $(H \cup G^*)$-bridges of G whose attachments are all contained in $V(qHs) \cup \{u_{t+1}\}$. Let $P_{t+1} = \emptyset$. We claim that

(6b) *there exists a qHs-Tutte path Q_{t+1} in J_{t+1} from u_{t+1} to s such that $q \in V(Q_{t+1})$.*

If $q = s$ then J_{t+1} is induced by the edge qu_{t+1}, and $Q_{t+1} := J_{t+1}$ is the desired path for (5b). So we may assume $q \neq s$. Then $J_{t+1} + su_{t+1}$ is 2-connected; and we may assume that it is a plane graph in which su_{t+1} and qHs are contained in a facial cycle. By applying Lemma 3.1, there is a qHs-Tutte path Q_{t+1} from u_{t+1} to s such that $q \in V(Q_{t+1})$. This concludes Step 6.

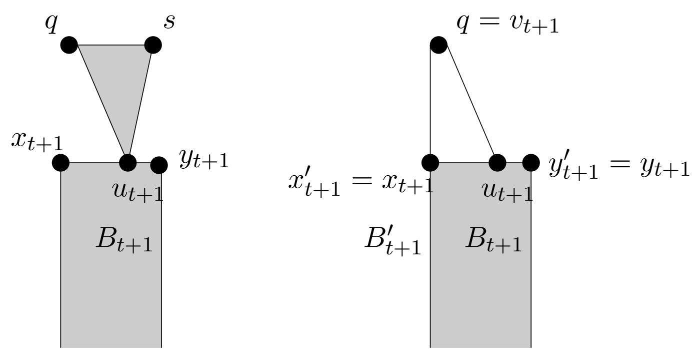


Figure 15: B'_{t+1} in Step 5.

We have now completed the definition of $B'_i, H'_i, u_i, v_i, w_i, x'_i, y'_i$ for all $1 \leq i \leq k$, with the possible exception: $i = t+1$, $w \in V(x_t H^* y_t - y_t)$, and $u' \in V(x_{t+1} H^* y_{t+1}) - \{x_{t+1}, y_{t+1}\}$. When this exceptional case occurs, we simply relabel (for each $i \in \{t+2, \ldots, k\}$) $B'_i, H'_i, u_i, v_i, w_i, x'_i, y'_i$ by $B'_{i-1}, H'_{i-1}, u_{i-1}, v_{i-1}, w_{i-1}, x'_{i-1}, y'_{i-1}$, respectively, and then relabel $k-1$ as k.

From the construction in Steps 1–6, we see that $V(B'_i) - V(B_i) = \{v_i\}$ for all $1 \leq i \leq k$. Since $G - V(\bigcup_{i=1}^k B_i)$ is finite, we see that $G - V(\bigcup_{i=1}^k B'_i)$ is finite. So the first part of (3) holds. It is clear from the definitions (and since G is $(4, H)$-connected) that the second part of (3) also holds.

We now prove (4). Let $B' := \bigcup_{i=1}^k B'_i$ and $H' := \bigcup_{i=1}^k H'_i$. Further, let X be a subgraph (finite or infinite) of B' containing H', and let T' be an H'-Tutte subgraph (finite or infinite) of X containing $\{v_i u_i, x'_i, y'_i : 1 \leq i \leq k\}$ such that each v_i has exactly one neighbor in $(T' \cap B'_i) - u_i$. Then each $T' \cap B'_i$ is an H'_i-Tutte subgraph of B'_i containing $\{v_i u_i, x'_i, y'_i\}$. So it follows from (1a), (2a), (2a′), (3a), (4a), (5a) and (6a) that $\{w_1, \ldots, w_k\} \subseteq V(T')$.

Let w'_i, u'_i denote the neighbors of v_i in $T' \cap B'_i$ (one of which is u_i) such that x'_i, w'_i, u'_i, y'_i occur on H'_i in order. We need to find an H-Tutte subgraph T of $G - V(B' - V(X))$ as in (4); and we do this by finding u'_{i-1}-w'_i paths (one for each $1 \leq i \leq k+1$, where $u'_0 = x$ and $w'_{k+1} = y$). We shall use an argument similar to that in the proof of Lemma 4.1. See Figure 16 for an illustration; noting that the paths between B_t and B_{t+1} change according to the location of w, u' and the path Q.)

First, we combine all paths found so far. Recall from Step 2 the path N_i in X_i from (2c) and (2c′); and for all other situations let us define $X_i = N_i = \emptyset$. Let $Y'_t := Y_t$ when $w \in V(y_t H^* x_{t+1} - x_{t+1})$; and otherwise define $Y'_t := \emptyset$. Recall the path L_t in Y_t from (c) and (d). Let $T^* := (T' - \{v_i : 1 \leq i \leq k\}) \cup L_t \cup (\bigcup_{i=1}^k N_i)$ and $X^* := (X - \{v_i : 1 \leq i \leq k\}) \cup (\bigcup_{i=1}^{k-1} (Y_i - Y'_t)) \cup (\bigcup_{i=1}^k X_i)$. Note that when $w \in V(y_t H^* x_{t+1} - x_{t+1})$, Y_t is not included in X^*, and $Q \cap Y_t$ is not included in T^*. Then $\{u'_i, w'_i, x'_i, y'_i : 1 \leq i \leq k\} \subseteq V(T^*)$. Moreover, every T^*-bridge of X^* is one of the following: Y_i for some $1 \leq i \leq k-1$, a T'-bridge of X containing no vertex from $\{v_i : 1 \leq i \leq k\}$, or a subgraph of X^* obtained from a T'-bridge of X by deleting its vertices in $\{v_i : 1 \leq i \leq k\}$, or an $(L_t + x_{t+1})$-bridge of Y_t when $w \in V(x_t H^* y_t - y_t)$ and $u' \in V(y_t H^* x_{t+1} - y_t)$ (see (c)), or an $(N_i + x_i)$-bridge of X_i for some $i \geq t+1$ (see (2c)), or an $(N_i + y_i)$-bridge of X_i for some $i \leq t-1$ (see (2c′)). Hence, it follows that T^* is an $(H^* \cap X^*)$-Tutte subgraph of X^*. (Note that $H^* \cap X^* = H^*$ if $w \in V(x_t H^* y_t - y_t)$; and otherwise $H^* \cap X^* = x_1 H^* y_t \cup x_{t+1} H^* y_k$.)

Recall that W denotes the set of attachments on H^* of all $(H \cup G^*)$-bridges of G. We

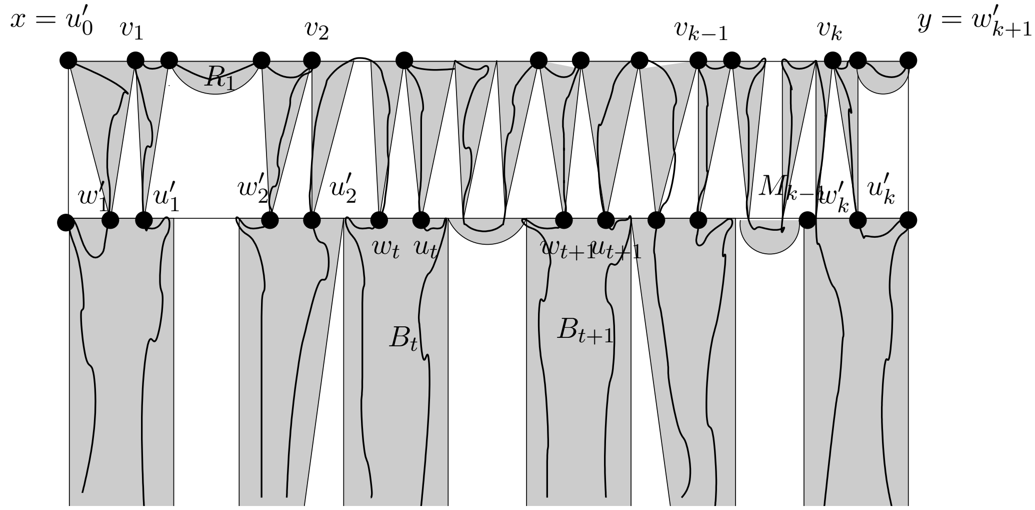


Figure 16: An illustration of T^* and T when $w, u' \in V(x_t H^* y_t)$.

define an equivalence relation $\sim$ on W as follows. For $z, z' \in W$, define $z \sim z'$ if $z = z'$, or $\{z, z'\} \subseteq V(D) - V(T^*)$ for some T^*-bridge D of X^* (in that case, D has just two attachments), or $\{z, z'\} \subseteq V(Y_t) - \{y_t, x_{t+1}\}$ when $w \in V(y_t H^* x_{t+1} - x_{t+1})$. Let $W_1, W_2, \ldots, W_m$ denote the equivalence classes of W with respect to $\sim$, and assume that $W_1, W_2, \ldots, W_m$ occur on H^* in order from x' to y'. Since $x', y' \in V(T^*)$, $W_1 = \{x'\}$ and $W_m = \{y'\}$. Note that either $|W_j| = 1$ and $W_j \subseteq V(T^*)$ (in which case let $D_j = W_j$), or $W_j \subseteq V(D_j) - V(T^*)$ for some T^*-bridge D_j of X^* (with just two attachments on T^*), or $W_j \subseteq V(Y_t) - \{y_t, x_{t+1}\}$ when $w \in V(y_t H^* x_{t+1} - x_{t+1})$ (in which case $D_j = Y_t$).

For each $1 \leq j \leq m$, let $s_j, t_j \in V(H)$ with $s_j H t_j$ maximal such that x, s_j, t_j, y occur on H in order, and s_j and t_j each are contained in an $(H \cup G^*)$-bridge of G with an attachment in W_j. Note from planarity that $x = s_1, t_1, s_2, t_2, \ldots, s_m, t_m = y$ occur on H in order. Also note that if $w \in V(y_t H^* x_{t+1} - x_{t+1})$ and $e \notin E(Q)$ (see (b)) then $e \in E(t_{h-1} H s_h)$ with $s_h = p$ for some $1 \leq h \leq m$; and if $w \in V(x_t H^* y_t - y_t)$ then $w_t = w$ and there is some $1 \leq h \leq m$ such that $W_h = \{w_t\}$ (because $w_t \in V(T')$), and $e \in E(t_{h-1} H t_h)$ (by the definition of w_t and u_t).

For each $1 \leq i \leq m-1$, let I_i denote the union of $t_i H s_{i+1}$ and those $(H \cup G^*)$-bridges of G whose attachments are all contained in $V(t_i H s_{i+1})$. (It is possible that $I_i \subseteq Y$ when $w \in V(y_t H^* x_{t+1} - x_{t+1})$.) For each $1 \leq j \leq m$, let U_j denote the union of $s_j H t_j$, D_j, and those $(H \cup G^*)$-bridges of G whose attachments are all contained in $V(s_j H t_j) \cup W_j$. Note that $|V(U_j \cap T^*)| = |V(D_j \cap T^*)| \leq 2$ and possibly $D_j = Y_t$ and $U_j \subseteq Y$. Recall that P_l, Q_l, J_l $(1 \leq l \leq k)$ are defined in (a), (1b), (2b), (2b′) and (6b).

By exactly the same argument as for Claims 1 and 2 in the proof of Lemma 4.1, we have the following two claims.

Claim 1. For each $1 \leq i \leq m-1$ *for which* $I_i \not\subseteq Y$ *and* $I_i \not\subseteq J_l$ $(1 \leq l \leq k)$*, there is a* $t_i H s_{i+1}$*-Tutte path* R_i *in* I_i *from* t_i *to* s_{i+1} *and containing* e *when* $e \in E(t_i H s_{i+1})$*.*

Claim 2. For each $1 \leq j \leq m$ *for which* $U_j \not\subseteq Y$ *and* $U_j \not\subseteq J_l$ $(1 \leq l \leq k)$*,* $U_j - V(U_j \cap T^*)$ *contains an* s_j*-*t_j *path* M_j *such that* $M_j \cup (U_j \cap T^*)$ *is an* $s_j H t_j$*-Tutte subgraph of* U_j*.*

By the choice of w, u', we know that $e \notin E(U_j)$ for all U_j with $U_j \not\subseteq Y$ and $U_j \not\subseteq J_l$ $(1 \leq l \leq k)$.

Let S denote the union of $P_l \cup Q_l \cup N_l$ $(1 \leq l \leq k)$, $Q \cap Y$ (when $w \in V(y_t H^* x_{t+1} - x_{t+1})$), R_i $(1 \leq i \leq m-1$ for which $I_i \not\subseteq Y$ and $I_i \not\subseteq J_l$ for all $1 \leq l \leq k)$, and all M_j $(1 \leq j \leq m$ for which $U_j \not\subseteq Y$ and $U_j \not\subseteq J_l$ for all $1 \leq l \leq k)$. From construction, each component of S is a u'_{n-1}-w'_n path $(1 \leq n \leq k)$, where $u'_0 = x$ and $w'_{k+1} = y$. See Figure 16. Note that the paths P_n, Q_n are from $\{w'_n, u'_n\}$ to $\{p_n, q_n\}$, and $\{p_n, q_n : 1 \leq n \leq k\} \subseteq \{s_j, t_j : 1 \leq j \leq m\}$. Also note that if $i \geq t+1$ then Q_{n-1} is from $u'_{n-1} = u_{n-1}$ to q_{n-1}, P_n is from $w'_n = w_n$ to p_n, and Q_{n-1}, P_n and some R_i's and M'_j form a u'_{n-1}-w'_n path; if $i \leq t-1$ then P_{n-1} is from $u'_{n-1} = w_{n-1}$ to p_{n-1}, Q_n is from $w'_n = u_n$ to q_n, and P_{n-1}, Q_n and some R_i's and M'_j form a u'_{n-1}-w'_n path.

Let $T := T^* \cup S$. From construction, $T' - \{v_i : 1 \leq i \leq k\} \subseteq T^* \subseteq T$, and $T - V(T' - \{u'_i, v_i, w'_i : 1 \leq i \leq k\}) = S$ which consists of disjoint u'_{n-1}-w'_n paths. For each $z \in V(T) - V(T')$, we see that $z \in V(S)$. Hence, either $z \notin V(X)$ or $z \in V(Z)$ for some T'-bridge Z of X containing an edge of H'. Thus (i), (ii) and (iii) of (4) hold.

To complete the proof of this lemma, we need to show that T is an H-Tutte subgraph of $G - V(B' - V(X))$. Let D be a T-bridge of $G - V(B' - V(X))$. Then D is induced by an edge of $G - E(X)$ with both incident vertices in X, or a T^*-bridge of X^* containing no edge of H^*, or a $((P_i \cup Q_i) + w_i)$-bridge of some J_i (see (a), (1a), (1b), (2a), (2b), (2a′), and (2b′)), or a Q_{t+1}-bridge of J_{t+1} (see (6b)), or a $(Q + y_t)$-bridge of $Y + qx_{t+1}$ when $w \in V(y_t H^* x_{t+1} - x_{t+1})$ (see (b)), or an R_i-bridge of I_i for some $1 \leq i \leq m-1$ with $I_i \not\subseteq Y$ and $I_i \not\subseteq J_l$ $(1 \leq l \leq k)$, or an $(M_j \cup (U_j \cap T^*))$-bridge of U_j for some $1 \leq j \leq m$ with $U_j \not\subseteq Y$ and $U_j \not\subseteq J_l$ $(1 \leq l \leq k)$. Thus, D has at most three attachments on T, and if D contains an edge of H then it has just two attachments on T. ∎

5 Hamilton circles

In this section, we prove a result which implies Theorem 1.4. First, we state a variation of the König Infinity Lemma, which is proved in [20]. We say that a sequence of finite paths $\{P_n\}$ *converge* to a ray P if, for any given $u, v \in V(P)$, $uPv = uP_n v$ for all sufficiently large n. (It is possible that a sequence converging to a ray may also converge to a different ray; but the sequences we consider will have a "forward" property which ensures a unique limit.)

Lemma 5.1 *Let G be an infinite locally finite graph and let $x \in V(G)$. Suppose $\{P_n\}$ is an infinite sequence of finite paths from x such that for all $n \geq 1$, the length of P_n increases. Then $\{P_n\}$ has an infinite subsequence $\{P_{n_k}\}$ converging to a ray from x.*

We also need two results from [2]. Let G be a locally finite graph. For a subgraph H of G and an end x of G, the *degree* of x in H is defined as $\sup\{|\mathcal{R}| : \mathcal{R}$ is a set of edge-disjoint x-arcs that are contained in the closure of $H\}$. (Note that an x-arc is an arc in x.) Bruhn and Stein [2] proved the following two results.

Lemma 5.2 *Let G be a locally finite graph, let $H \subseteq G$, and let x be an end of G. Then for any natural number k, the degree of x in H is k if and only if k is smallest integer such that every finite $S \subseteq V(G)$ can be separated from x with a finite edge cut in G that shares exactly k edges with $E(H)$.*

Lemma 5.3 *Let C be a subgraph of a locally finite graph G. Then the closure of C is a circle in $|G|$ iff it is topologically connected and every vertex or end x of G contained in the closure of C has degree 2 in C.*

In later proofs, we need to find finite paths which converge to double rays whose union is a Tutte subgraph. For this reason, we need those finite paths to move towards the ends of the graph. Let $(H_1, H_2, \ldots)$ be a sequence (finite or infinite) of subgraphs of an infinite graph G. A path P in G is $(H_1, H_2, \ldots)$-*forward* if, for every $i \geq 1$ and for any distinct $x, y, z \in V(P)$ with $y \in V(xPz)$, $\{x, z\} \subseteq V(H_i)$ implies that $y \notin V(H_j)$ for all $j \geq i+2$. Intuitively, if P starts from H_1, then after P meets H_{i+2} (for each $i \geq 1$), P never visits H_i again.

Lemma 5.4 *Let (G, H, x, y) be a 4-tuple, and assume that G is $(4, H)$-connected. Then for any $e \in E(H)$, there is an H-Tutte subgraph T of G such that $e \in E(T)$ and the closure of T in $|G|$ is an arc between x and y.*

Proof. First, we construct an infinite sequence of collections: $\{(B_i^j, H_i^j, e_i^j, x_i^j, y_i^j) : 1 \leq i \leq k_j\}$ for $j \geq 1$, where each $(B_i^j, H_i^j, x_i^j, y_i^j)$ is a 4-tuple and $e_i^j \in E(H_i^j)$.

Let $k_1 = 1, B_1^1 := G$, $H_1^1 := H$, $e_1^1 := e$, $x_1^1 := x$, and $y_1^1 := y$. Suppose we have constructed $\{(B_i^j, H_i^j, e_i^j, x_i^j, y_i^j) : 1 \leq i \leq k_j\}$ for some odd integer $j \geq 1$ such that $B_i^j \cap B_l^j = \emptyset$ for $1 \leq i < l \leq k_j$ (except possibly $V(B_i^j \cap B_{i+1}^j) = \{y_i^j = x_{i+1}^j\}$), each $(B_i^j, H_i^j, x_i^j, y_i^j)$ is a 4-tuple, B_i^j is $(4, H_i^j)$-connected, and $e_i^j \in E(H_i^j)$. Let $B^j := \bigcup_{i=1}^{k_j} B_i^j$ and $H^j := \bigcup_{i=1}^{k_j} H_i^j$. Note that $B^1 = G$ and $H^1 = H$.

By applying Lemma 4.2 to each B_t^j, $1 \leq t \leq k_j$, and by an appropriate labeling (say, from left to right according to the embedding), there exist 4-tuples $(B_i^{j+1}, H_i^{j+1}, x_i^{j+1}, y_i^{j+1})$ $(1 \leq i \leq k_{j+1})$ and there exist paths $u_i^{j+1} v_i^{j+1} w_i^{j+1}$ on H_i^{j+1} such that

(1) B_i^{j+1} is $(4, H_i^{j+1})$-connected, $V(H_i^{j+1} \cap H^j) = \{v_i^{j+1}\}$, and B_i^{j+1} is an induced subgraph of $(B^j - V(H^j - v_i^{j+1})) + \{v_i^{j+1} u_i^{j+1}, v_i^{j+1} w_i^{j+1}\}$;

(2) no edge of B^j joins a vertex of $B_i^{j+1} - V(H_i^{j+1})$ to a vertex of $B^j - V(B_i^{j+1})$;

(3) $B^j - V(B^{j+1})$ is finite, where $B^{j+1} := \bigcup_{i=1}^{k_{j+1}} B_i^{j+1}$, and $B_i^{j+1} \cap B_l^{j+1} = \emptyset$ for all $1 \leq i < l \leq k_{j+1}$ unless $l = i+1$, $v_i^{j+1} = v_{i+1}^{j+1}$, and $V(B_i^{j+1} \cap B_{i+1}^{j+1}) = \{v_i^{j+1} = v_{i+1}^{j+1}\}$;

(4) for any subgraph (finite or infinite) X of B^{j+1} containing $H^{j+1} := \bigcup_{i=1}^{k_{j+1}} H_i^{j+1}$, and for any H^{j+1}-Tutte subgraph (finite or infinite) T^{j+1} of X containing $\{v_i^{j+1} u_i^{j+1}, x_i^{j+1}, y_i^{j+1} : 1 \leq i \leq k_{j+1}\}$ such that each v_i^{j+1} has exactly one neighbor in $(T^{j+1} \cap B_i^{j+1}) - u_i^{j+1}$, there exists an H^j-Tutte subgraph T^j in $B^j - V(B^{j+1} - V(X))$ containing $\{e_i^j : 1 \leq i \leq k_j\}$ such that

   (i) $T^{j+1} - \{v_i^{j+1} : 1 \leq i \leq k_{j+1}\} \subseteq T^j$,

   (ii) for any $z \in V(T^j) - V(T^{j+1})$, either $z \notin V(X)$, or $z \in V(Z)$ for some T^{j+1}-bridge Z of X containing an edge of H^{j+1}, and

   (iii) if z_i^{j+1}, a_i^{j+1} (corresponding to w_i', u_i' in Lemma 4.2) denote the neighbors of v_i^{j+1} in $T^{j+1} \cap B_i^{j+1}$ such that $x_i^{j+1}, z_i^{j+1}, a_i^{j+1}, y_i^{j+1}$ occur on H_i^{j+1} in order, then $T^j - V(T^{j+1} - \{v_i^{j+1}, a_i^{j+1}, z_i^{j+1} : 1 \leq i \leq k_{j+1}\})$ consists of the following disjoint paths: one from a_s^{j+1} to z_{s+1}^{j+1} (for each pair $\{a_s^{j+1}, z_{s+1}^{j+1}\}$ that is contained in B_i^j for some $1 \leq i \leq k_j$), one from x_i^j to z_{s+1}^{j+1} (with $s = 0$ and $i = 1$, or for each pair $\{a_s^{j+1}, z_{s+1}^{j+1}\}$ with $s > 1$ and for some $1 \leq i \leq k_j$, $a_s^{j+1} \notin V(B_i^j)$ and $z_{s+1}^{j+1} \in V(B_i^j)$), and one from a_s^{j+1} to y_i^j (with $s = k_{j+1}$ and $i = k_j$, or for each pair $\{a_s^{j+1}, z_{s+1}^{j+1}\}$ with $s < k_{j+1}$ and for some $1 \leq i \leq k_j$, $a_s^{j+1} \in V(B_i^j)$ and $z_{s+1}^{j+1} \notin V(B_i^j)$).

Let $e_i^{j+1} = v_i^{j+1} u_i^{j+1}$, $1 \le i \le k_{j+1}$. By applying Lemma 4.1 to each B_t^{j+1} ($1 \le t \le k_{j+1}$), and by an appropriate labeling (from left to right according to the embedding of G), there exist 4-tuples $(B_i^{j+2}, H_i^{j+2}, x_i^{j+2}, y_i^{j+2})$ ($1 \le i \le k_{j+2}$) and there exist $e_i^{j+2} \in E(H_i^{j+2})$ such that

(5) B_i^{j+2} is an induced subgraph of $B^{j+1} - V(H^{j+1})$, and B_i^{j+2} is $(4, H_i^{j+2})$-connected;

(6) no edge of B^{j+1} joins a vertex of $B_i^{j+2} - V(H_i^{j+2})$ to a vertex of $B^{j+1} - V(B_i^{j+2})$;

(7) $B^{j+1} - V(B^{j+2})$ is finite, where $B^{j+2} := \bigcup_{i=1}^{k_{j+2}} B_i^{j+2}$, and $B_i^{j+2} \cap B_l^{j+2} = \emptyset$ for all $1 \le i < l \le k_{j+2}$ unless $l = i+1$, $y_i^{j+2} = x_{i+1}^{j+2}$, and $V(B_i^{j+2} \cap B_{i+1}^{j+2}) = \{y_i^{j+2} = x_{i+1}^{j+2}\}$;

(8) for any subgraph (finite or infinite) X of B^{j+2} containing $H^{j+2} := \bigcup_{i=1}^{k_{j+2}} H_i^{j+2}$, and for any H^{j+2}-Tutte subgraph (finite or infinite) T^{j+2} of X containing $\{e_i^{j+2}, x_i^{j+2}, y_i^{j+2} : 1 \le i \le k_{j+2}\}$, there exists an H^{j+1}-Tutte subgraph T^{j+1} of $B^{j+1} - (V(B^{j+2}) - V(X))$ containing $\{e_i^{j+1}, x_i^{j+1}, y_i^{j+1} : 1 \le i \le k_{j+1}\}$ such that

   (i) $T^{j+2} \subseteq T^{j+1}$,

   (ii) for any $z \in V(T^{j+1}) - V(T^{j+2})$, either $z \notin V(X)$, or $z \in V(Z)$ for some T^{j+2}-bridge Z of X containing an edge of H^{j+2}, and

   (iii) $T^{j+1} - V(T^{j+2} - \{x_i^{j+2}, y_i^{j+2} : 1 \le i \le k_{j+2}\})$ consists of the following disjoint paths: one from y_i^{j+2} to x_{i+1}^{j+2} (for each pair $\{y_i^{j+2}, x_{i+1}^{j+2}\}$ that is contained in B_t^{j+1} for some $1 \le t \le k_{j+1}$), and one from x_p^{j+2} to y_q^{j+2} (for each pair $\{x_p^{j+2}, y_q^{j+2}\}$ such that there exists some $1 \le t \le k_{j+1}$ for which $x_p^{j+2}, y_q^{j+2} \in V(B_t^{j+1})$ and, subject to this, p is minimum and q is maximum).

Note that $V(H^j \cap H^{j+1}) = \{v_i^{j+1} : 1 \le i \le k_{j+1}\}$ when j is odd (see (1)), and $H^j \cap H^{j+1} = \emptyset$ when j is even (see (5)). Hence $H^j \cap H^{j+2} = \emptyset$ for $j \ge 1$. Also note that because of (2) and (6), $V(H^j)$ is a cut set of G for $j \ge 2$.

For any $n \ge j \ge 1$, let $B_{j,n} := B^j - (V(B^n) - V(H^n))$. Note that $B_{j,n}$ is a subgraph of B^j. In order to find the desired T, we need to find a sequence of subgraphs of $B_{1,n}$ for $n \ge 1$ which converge to T. These subgraphs are found in (9) below. But we need to prove (9) and (10) simultaneously.

(9) When j is odd, there is an H^j-Tutte subgraph $T_{j,n}$ of $B_{j,n}$ containing $\{e_i^j, x_i^j, y_i^j : 1 \le i \le k_j\}$, such that the components of each $T_{j,n} \cap B_t^j$ ($1 \le t \le k_j$) are paths: one $(H^j, \ldots, H^n)$-forward path from x_t^j to H^n, one $(H^j, \ldots, H^n)$-forward path from y_t^j to H^n, and one path between two vertices of H^n and containing $\{y_i^s, x_{i+1}^s\}$ (for each ordered pair (s, i), $j+1 \le s \le n$ and $1 \le i \le k_s - 1$, with $\{y_i^s, x_{i+1}^s\}$ contained in both B_t^j and a component of B^{s-1}) in which the disjoint paths from y_i^s, x_{i+1}^s to H^n are $(H^j, \ldots, H^n)$-forward. Moreover, for each $1 \le t \le k_j$, there is some finite path $P_{j,n}^t$ in B_t^j from x_t^j to y_t^j such that $T_{j,n} \cap B_t^j \subseteq P_{j,n}^t$ and $(P_{j,n}^t - V(T_{j,n})) \cap H^j = \emptyset$.

(10) When j is even, there is an H^j-Tutte subgraph $T_{j,n}$ in $B_{j,n}$ containing $\{e_i^j := v_i^j u_i^j, x_i^j, y_i^j : 1 \le i \le k_j\}$, such that the components of each $T_{j,n} \cap B_t^j$ ($1 \le t \le k_j$) are paths between two vertices of H^n: one containing $\{e_t^j = v_t^j u_t^j, x_t^j, y_t^j\}$ in which the disjoint paths from v_t^j, u_t^j to H^n are $(H^j, \ldots, H^n)$-forward, and one containing $\{y_i^s, x_{i+1}^s\}$ (for each ordered pair (s, i),

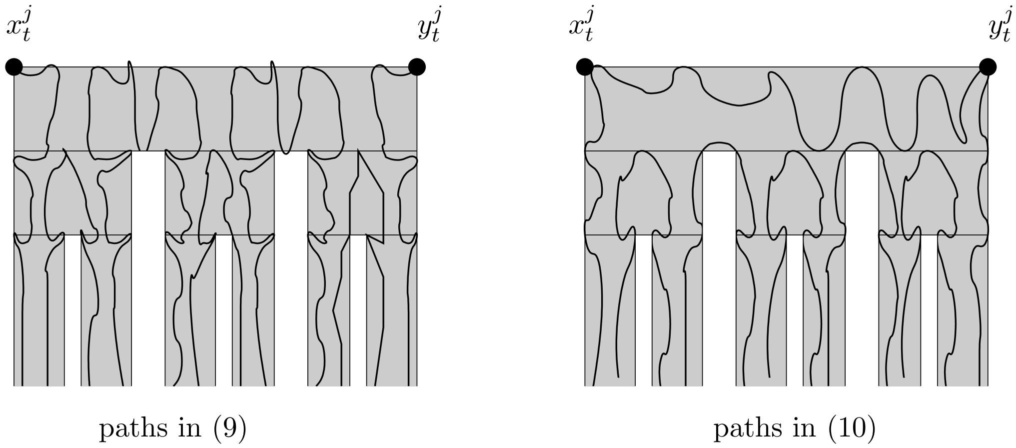


Figure 17: Illustrations of paths in (9) and (10).

$j+1 \le s \le n$ and $1 \le i \le k_s - 1$, with $\{y_i^s, x_{i+1}^s\}$ contained in both B_t^j and a component of B^{s-1}) in which the disjoint paths from y_i^s, x_{i+1}^s to H^n are $(H^j, \ldots, H^n)$-forward. Moreover, for each $1 \le t \le k_j$, there is some finite cycle $C_{j,n}^t$ in B_t^j such that $T_{j,n} \cap B_t^j \subseteq C_{j,n}^t$ and $(C_{j,n}^t - V(T_{j,n})) \cap H^j = \emptyset$.

We apply induction on $n-j$. Note that $n-j \ge 0$. When $n-j=0$, we have $B_{j,n} = H^j$, and we simply let $T_{j,n} := H^j$. Then for each $1 \le t \le k_j$, $T_{j,n} \cap B_t^j = H^j \cap B_t^j = H_t^j$ is a finite path in G' from x_t^j to y_t^j. So when j is odd, we may take $P_{j,n}^t = H_t^j$; and we see that $T_{j,n}$ is the desired H^j-Tutte subgraph for (9). Now assume j is even. Then since B_t^j is 2-connected and planar, H_t^j is contained in some cycle, say $C_{j,n}^t$, of B_t^j; and we see that $T_{j,n}$ gives the desired H^j-Tutte subgraph for (10). Hence we may assume $n-j \ge 1$.

To prove (9), suppose j is odd. Then $j+1$ is even, and we apply (10) to $B_{j+1,n}$ inductively. So there is an H^{j+1}-Tutte subgraph $T_{j+1,n}$ of $B_{j+1,n}$ containing $\{e_i^{j+1} := v_i^{j+1}u_i^{j+1}, x_i^{j+1}, y_i^{j+1} : 1 \le i \le k_{j+1}\}$, such that the components of each $T_{j+1,n} \cap B_t^{j+1}$ $(1 \le t \le k_{j+1})$ are paths between two vertices of H^n: one containing $\{e_t^{j+1} = v_t^{j+1}u_t^{j+1}, x_t^{j+1}, y_t^{j+1}\}$ in which the disjoint paths from v_t^{j+1}, u_t^{j+1} to H^n are $(H^{j+1}, \ldots, H^n)$-forward, and one containing $\{y_i^s, x_{i+1}^s\}$ (for each ordered pair (s,i), $j+2 \le s \le n$ and $1 \le i \le k_s - 1$, with $\{y_i^s, x_{i+1}^s\}$ contained in both B_t^{j+1} and a component of B^{s-1}) in which the disjoint paths from y_i^s, x_{i+1}^s to H^n are $(H^{j+1}, \ldots, H^n)$-forward. Moreover, for each $1 \le t \le k_{j+1}$, there is some finite cycle $C_{j+1,n}^t$ in B_t^{j+1} such that $T_{j+1,n} \cap B_t^{j+1} \subseteq C_{j+1,n}^t$ and $(C_{j+1,n}^t - V(T_{j+1,n})) \cap H^{j+1} = \emptyset$.

For $1 \le i \le k_{j+1}$, let z_i^{j+1}, a_i^{j+1} denote the neighbors of v_i^{j+1} in $T_{j+1,n} \cap B_i^{j+1}$ such that $x_i^{j+1}, z_i^{j+1}, a_i^{j+1}, y_i^{j+1}$ occur on H_i^{j+1} in order. By applying (4) with $X = B_{j+1,n}$, there exists an H^j-Tutte subgraph $T_{j,n}$ of $B_{j,n}$ containing $\{e_i^j, x_i^j, y_i^j : 1 \le i \le k_j\}$ such that

(i) $T_{j+1,n} - \{v_i^{j+1} : 1 \le i \le k_{j+1}\} \subseteq T_{j,n}$,

(ii) for any $z \in V(T_{j,n}) - V(T_{j+1,n})$, either $z \notin V(B_{j+1,n})$, or $z \in V(Z)$ for some $T_{j+1,n}$-bridge Z of $B_{j+1,n}$ containing an edge of H^{j+1}, and

(iii) $T_{j,n} - V(T_{j+1,n} - \{v_i^{j+1}, a_i^{j+1}, z_i^{j+1} : 1 \le i \le k_{j+1}\})$ consists of the following disjoint paths: one from a_s^{j+1} to z_{s+1}^{j+1} (for each pair $\{a_s^{j+1}, z_{s+1}^{j+1}\}$ contained in some B_i^j, $1 \le i \le k_j$), one from x_i^j to z_{s+1}^{j+1} (with $s = 0$ and $i = 1$, or for each pair $\{a_s^{j+1}, z_{s+1}^{j+1}\}$ with $s > 1$ and for some B_i^j, $a_s^{j+1} \notin V(B_i^j)$ and $z_{s+1}^{j+1} \in V(B_i^j)$), and one path from a_s^{j+1} to y_i^j (with $s = k_{j+1}$ and $i = k_j$, or for each pair $\{a_s^{j+1}, z_{s+1}^{j+1}\}$ with $s < k_{j+1}$ and for some B_i^j, $a_s^{j+1} \in V(B_i^j)$ and $z_{s+1}^{j+1} \notin V(B_i^j)$).

By (iii), for each $1 \le t \le k_j$, the components of $T_{j,n} \cap B_t^j$ are the following paths: one from x_t^j to H^n, one from y_t^j to H^n, and one between two vertices of H^n and containing $\{y_i^s, x_{i+1}^s\}$ (for each ordered pair (s, i), $j + 1 \le s \le n$ and $1 \le i \le k_s - 1$, with $\{y_i^s, x_{i+1}^s\}$ contained in both B_t^j and a component of B^{s-1}). Because of $C_{j+1,n}^l$ for $1 \le l \le k_{j+1}$, we see that for each $1 \le t \le k_j$, there exists some finite path $P_{j,n}^t$ in B_t^j such that $T_{j,n} \cap B_t^j \subseteq P_{j,n}^t$ and $(P_{j,n}^t - V(T_{j,n})) \cap H^j = \emptyset$.

It remains to prove the forwardness of paths in $T_{j,n} \cap B_t^j$, for $1 \le t \le k_j$. Note that any component C of $T_{j,n} \cap B_t^j$ containing $\{y_i^s, x_{i+1}^s\}$ for some (s, i) (with $j + 2 \le s \le n$ and $1 \le i \le k_s - 1$) is also a component of $T_{j+1,n} \cap B_r^{j+1}$ for some $1 \le r \le k_{j+1}$. Hence by induction hypothesis, the disjoint paths in C from y_i^s, x_{i+1}^s to H^n are $(H^{j+1}, \ldots, H^n)$-forward, and thus also $(H^j, \ldots, H^n)$-forward since C is disjoint from H^j.

Let D be a component of $T_{j,n} \cap B_t^j$ containing $\{y_i^{j+1}, x_{i+1}^{j+1}\}$ (for some $1 \le i \le k_{j+1} - 1$), and let D_y, D_x be the disjoint paths in D from y_i^{j+1}, x_{i+1}^{j+1} to H^n, respectively. Then D_y (respectively, D_x) is contained in one of the disjoint paths in the component of $T_{j+1,n} \cap B_i^{j+1}$ (respectively, $T_{j+1,n} \cap B_{i+1}^{j+1}$) containing $\{x_i^{j+1}, y_i^{j+1}, v_i^{i+1}u_i^{j+1}\}$ (respectively, $\{x_{i+1}^{j+1}, y_{i+1}^{j+1}, v_{i+1}^{i+1}u_{i+1}^{j+1}\}$) from u_i^{j+1} or v_{i+1}^{j+1} to H^n. Hence, by induction hypothesis, D_y and D_x are $(H^{j+1}, \ldots, H^n)$-forward, and thus also $(H^j, \ldots, H^n)$-forward since D_x and D_y are disjoint from H^j.

To complete the proof of (9), we need to show that the paths L_x, L_y in $T_{j,n} \cap B_t^j$ from x_t^j, y_t^j to H^n are $(H^j, \ldots, H^n)$-forward. Let k be minimum such that $z_k^{j+1} \in V(B_{j+1,n} \cap B_t^j)$ and let l be maximum such that $a_l^{j+1} \in V(B_{j+1,n} \cap B_t^j)$. Let L_z (respectively, L_a) denote the path in $T_{j+1,n} \cap B_k^{j+1}$ (respectively, $T_{j+1,n} \cap B_l^{j+1}$) from z_k^{j+1} (respectively, a_l^{j+1}) to H^n which is contained in L_x (respectively, L_y). By induction hypothesis, L_z, L_a are $(H^{j+1}, \ldots, H^n)$-forward (and thus $(H^j, \ldots, H^n)$-forward since L_z and L_a are disjoint from H^j). We now prove that L_x is $(H^j, \ldots, H^n)$-forward. Let $a, b, c \in V(L_x)$ such that $b \in V(aL_xc)$. Suppose $a, c \in V(H^m)$ for some $j \le m \le n$. We show next that $b \notin V(H^r)$ for all $r \ge m + 2$. This is clear when $a, c \in V(L_z)$ since L_z is $(H^j, \ldots, H^n)$-forward. Now assume $a, c \in V(L_x) - V(L_z - z_k^{j+1})$. Then since $H^{j+2} \cap H^{j+1} = \emptyset$ (because $j + 1$ is even) and by (ii), we have $aL_xc \cap H^{j+2} = \emptyset$, and hence $b \notin V(H^r)$ for all $r \ge m + 2 \ge j + 2$. So we may assume by symmetry that $a \in V(L_x) - V(L_z)$ and $c \in V(L_z - z_k^{j+1})$. Then $m = j + 1$. If $b \in V(L_z)$ then $b \in V(z_k^{j+1}L_zc)$; and since $\{z_k^{j+1}, c\} \subseteq V(H^{j+1})$, $b \notin V(H^r)$ for $r \ge m + 2 = j + 3$ (since L_z is $(H^j, \ldots, H^n)$-forward). So we may assume $b \notin V(L_z)$. Then again, since $H^{j+2} \cap H^{j+1} = \emptyset$ and by (ii), $b \notin V(H^r)$ for all $r \ge m + 2$. This shows that L_x is $(H^j, \ldots, H^n)$-forward. The same argument (with L_y, L_a, a_l^{j+1} playing the roles of L_x, L_z, z_k^{j+1}, respectively) shows that L_y is also $(H^j, \ldots, H^n)$-forward.

We now prove (10). Suppose j is even. Then $j + 1$ is odd. By inductively applying (9), there is an H^{j+1}-Tutte subgraph $T_{j+1,n}$ of $B_{j+1,n}$ containing $\{e_i^{j+1}, x_i^{j+1}, y_i^{j+1} : 1 \le i \le k_{j+1}\}$, such that the components of each $T_{j+1,n} \cap B_t^{j+1}$ $(1 \le t \le k_{j+1})$ are the following paths: one $(H^{j+1}, \ldots, H^n)$-forward path from x_t^{j+1} to H^n, one $(H^{j+1}, \ldots, H^n)$-forward path from y_t^{j+1} to

H^n, and one between vertices of H^n and containing $\{y_i^s, x_{i+1}^s\}$ (for each ordered pair (s,i), $j+2 \le s \le n$ and $1 \le i \le k_s - 1$, with $\{y_i^s, x_{i+1}^s\}$ contained in both B_t^{j+1} and a component of B^{s-1}) in which the disjoint paths from y_i^s, x_{i+1}^s to H^n are $(H^{j+1}, \ldots, H^n)$-forward. Moreover, for each $1 \le t \le k_{j+1}$, there is some finite path $P_{j+1,n}^t$ in B_t^{j+1} such that $T_{j+1,n} \cap B_t^{j+1} \subseteq P_{j+1,n}^t$ and $(P_{j+1,n}^t - V(T_{j+1,n})) \cap H^{j+1} = \emptyset$.

By applying (8) (with B^{j+1} as B^{j+2} and $X = B_{j+1,n}$), there exists an H^j-Tutte subgraph $T_{j,n}$ of $B_{j,n}$ containing $\{e_i^j, x_i^j, y_i^j : 1 \le i \le k_j\}$ such that

(i) $T_{j+1,n} \subseteq T_{j,n}$,

(ii) for any $z \in V(T_{j,n}) - V(T_{j+1,n})$, either $z \notin V(B_{j+1,n})$, or $z \in V(Z)$ for some $T_{j+1,n}$-bridge Z of $B_{j+1,n}$ containing an edge of H^{j+1}, and

(iii) $T_{j,n} - V(T_{j+1,n} - \{x_i^{j+1}, y_i^{j+1} : 1 \le i \le k_{j+1}\})$ consists of the following disjoint paths: one from y_i^{j+1} to x_{i+1}^{j+1} (for each pair $\{y_i^{j+1}, x_{i+1}^{j+1}\}$ contained in B_l^j for some $1 \le l \le k_j$), and one from x_p^{j+1} to y_q^{j+1} (for each pair $\{x_p^{j+1}, y_q^{j+1}\}$ such that there exists some $1 \le l \le k_j$ for which $x_p^{j+1}, y_q^{j+1} \in V(B_l^j)$ with p smallest and q largest).

By (iii), the components of each $T_{j,n} \cap B_t^j$ $(1 \le t \le k_j)$ are the following paths: one containing $\{e_t^j = v_t^j u_t^j, x_t^j, y_t^j\}$, and one between two vertices of H^n and containing $\{y_i^s, x_{i+1}^s\}$ (for each ordered pair (s,i), $j+1 \le s \le n$ and $1 \le i \le k_s - 1$, with $\{y_i^s, x_{i+1}^s\}$ contained in both B_t^j and a component of B^{s-1}). Because of $P_{j+1,n}^l$ for $1 \le l \le k_{j+1}$, we see that for each $1 \le t \le k_j$, there is some finite cycle $C_{j,n}^t$ in B_t^j such that $T_{j,n} \cap B_t^j \subseteq C_{j,n}^t$ and $(C_{j,n}^t - V(T_{j,n})) \cap H^j = \emptyset$.

We need to prove the forwardness of paths in $T_{j,n} \cap B_t^j$ for $1 \le t \le k_j$. Note that any component C of $T_{j,n} \cap B_t^j$ containing $\{y_i^s, x_{i+1}^s\}$ for some (s,i), $j+2 \le s \le n$ and $1 \le i \le k_s - 1$, is also a component of $T_{j+1,n} \cap B_l^{j+1}$ (for some $1 \le l \le k_{j+1}$). Hence by induction hypothesis, the disjoint paths in C from y_i^s, x_{i+1}^s to H^n are $(H^{j+1}, \ldots, H^n)$-forward, and thus also $(H^j, \ldots, H^n)$-forward since C is disjoint from H^j.

Let D be a component of $T_{j,n} \cap B_t^j$ containing $\{y_i^{j+1}, x_{i+1}^{j+1}\}$ (for some $1 \le i \le k_{j+1} - 1$), and let D_y, D_x be the disjoint paths in D from y_i^{j+1}, x_{i+1}^{j+1} to H^n, respectively. Then D_y (respectively, D_x) is a component of $T_{j+1,n} \cap B_i^{j+1}$ (respectively, $T_{j+1,n} \cap B_{i+1}^{j+1}$). Hence, by induction hypothesis, D_y and D_x are $(H^{j+1}, \ldots, H^n)$-forward, and thus $(H^j, \ldots, H^n)$-forward since they are disjoint from H^j.

To complete the proof of (10), we need to show that the paths L_v, L_u from v_t^j, u_t^j to H^n in the component of $T_{j,n} \cap B_t^j$ containing $\{e_t^j, x_t^j, y_t^j\}$ are $(H^j, \ldots, H^n)$-forward. Let k be minimum such that $x_k^{j+1} \in V(B_{j+1,n} \cap B_t^j)$ and let l be maximum such that $y_l^{j+1} \in V(B_{j+1,n} \cap B_t^j)$. Let L_x (respectively, L_y) denote the path in $T_{j+1,n} \cap B_k^{j+1}$ (respectively, $T_{j+1,n} \cap B_l^{j+1}$) from x_k^{j+1} (respectively, y_l^{j+1}) to H^n. Then by planarity, $L_x \subseteq L_v$ and $L_y \subseteq L_u$, or $L_x \subseteq L_u$ and $L_y \subseteq L_v$. We may assume the former; as the argument for the latter is exactly the same. By induction hypothesis, L_x, L_y are $(H^{j+1}, \ldots, H^n)$-forward (and thus $(H^j, \ldots, H^n)$-forward since they are disjoint from H^j). We now prove that L_v is $(H^j, \ldots, H^n)$-forward. Let $a, b, c \in V(L_v)$ such that $b \in V(aL_v c)$. Suppose $a, c \in V(H^m)$ for some $j \le m \le n$. We show next that $b \notin V(H^r)$ for all $r \ge m+2$. This is clear when $a, c \in V(L_x)$ since L_x is $(H^j, \ldots, H^n)$-forward. Now assume $a, c \in V(L_v) - V(L_x - x_k^{j+1})$. Then since $V(H^{j+2} \cap H^{j+1}) = \{v_i^{j+2} : 1 \le i \le k_{j+2}\}$ and by (ii), we have $aL_v c \cap H^{j+2} = \emptyset$. So $b \notin V(H^r)$ for all $r \ge m+2 \ge j+2$. We may thus assume by symmetry that $a \in V(L_v) - V(L_x)$ and $c \in V(L_x - x_k^{j+1})$. Then $m = j+1$. If $b \in V(L_x)$ then $b \in V(x_k^{j+1} L_x c)$; and since $\{x_k^{j+1}, c\} \subseteq V(H^{j+1})$, $b \notin V(H^r)$ for $r \geq m+2 = j+3$ (since L_x is $(H^j, \ldots, H^n)$-forward). So we may assume $b \notin V(L_x)$. Then $b \in V(aL_v x_k^{j+1}) \subseteq V(L_v) - V(L_x - x_k^{j+1})$, which, as above, implies $b \notin V(H^r)$ for all $r \geq m+2$. This shows that L_v is $(H^j, \ldots, H^n)$-forward. The same argument (with L_u, L_y, y_l^{j+1} playing the roles of L_v, L_x, x_k^{j+1}, respectively) shows that L_u is also $(H^j, \ldots, H^n)$-forward. This completes the inductive proof of (10).

Next, we show how to construct the desired graph T. For each $n \geq 1$, let $T_n := T_{1,n}$. Then T_n is a subgraph of G, because $T_n \subseteq B^1 = G$. From (9), the components of T_n are the following paths: a path X_n^1 from $x = x_1^1$ to H^n, a path Y_n^1 from $y = y_{k_1}^1$ to H^n, and one path $P_n^{s,i}$ between two vertices of H^n and containing $\{y_i^s, x_{i+1}^s\}$ (for each ordered pair (s,i), $2 \leq s \leq n$ and $1 \leq i \leq k_s - 1$, with $\{y_i^s, x_{i+1}^s\}$ contained in a component of B^{s-1}).

Since each $V(H^n)$ is a cut set of G, it follows from Lemma 5.1 that there is an infinite subsequence $\{X_{n_k}^1\}$ of $\{X_n^1\}$ converging to a ray X^1 from $x_1^1 = x$. Note that any subsequence of $\{X_{n_k}^1\}$ also converges to X^1. Similarly, since all $Y_{n_k}^1$ are $(H^1, H^2, \ldots)$-forward, it follows from Lemma 5.1 that there is an infinite subsequence $\{Y_{n_{1,1}}^1\}$ of $\{Y_{n_k}^1\}$ converging to a ray Y^1 from $y_1^1 = y$. Note that $\{X_{n_{1,1}}^1\}$ also converges to X^1.

For each $s \geq 2$ and $1 \leq i \leq k_s - 1$, the disjoint paths in $P_n^{s,i}$ from y_i^s, x_{i+1}^s to H^n are $(H^s, \cdots, H^n)$-forward. Hence, because $y_i^s P_n^{s,i} x_{i+1}^s \subseteq B^{s-1} - V(B^s - \{y_i^s, x_{i+1}^s\})$ which is finite, we can apply Lemma 5.1 to show that any infinite subsequence of $\{P_n^{s,i}\}$ contains an infinite subsequence that converges to a double ray, say $P^{s,i}$, containing $\{y_i^s, x_{i+1}^s\}$.

Consider the lexicographic ordering of $\{(s,i) : s \geq 1, 1 \leq i \leq k_s - 1\}$; so $(s,i) < (t,j)$ if, and only if, $s < t$, or $s = t$ and $i < j$. We choose sequences $\{n_{s,i}\}$ such that $\{P_{n_{s,i}}^{s,i}\}$ converges to a double ray $P^{s,i}$; and if $(s,i) < (t,j)$ then $\{n_{t,j}\}$ is a subsequence of $\{n_{s,i}\}$. Note that for any $(s,i) \neq (t,j)$ (where $s,t \geq 2$, $1 \leq i \leq k_s - 1$, and $1 \leq j \leq k_t - 1$), $P^{s,i}$ and $P^{t,j}$ (when both defined) are disjoint. Also note that the two ends of $P^{s,i}$ must belong to distinct ends of G.

Let T denote the union of X^1, Y^1 and all $P^{s,i}$. Note that $T \subseteq G$, and each component of T other than X^1, Y^1 is a double ray $P^{s,i}$ for some (s,i). Also note that each $T_{1,n} \cap B_1^1 = T_{1,n} \cap G$ is contained in the finite path $P_{1,n}^1$ (see (9)) of G between x and y.

We wish to apply Lemma 5.3 to show that the closure of $T + xy$ is a circle. Because of the paths in H^i ($H^i \subseteq G$ for all odd i), we see that no finite cut of G disjoint from T can separate T, and hence the closure of $T + xy$ is topologically connected. Furthermore, for any end z of G and any finite $S \subseteq V(G)$, there is finite edge cut of G separating z from S and intersecting $E(T)$ exactly twice, and any finite edge cut of G separating z from S cannot intersect $E(T)$ exactly once. So by Lemma 5.2, we see that each end of G has degree 2 or 0 in T. Thus by applying Lemma 5.3 to $T + xy$ and $G + xy$, we see that the closure of $T + xy$ is a circle. So

(11) the closure of T in $|G|$ is an arc between x and y.

It remains to show that T is an H-Tutte subgraph of G. Let D be a T-bridge of G.

We claim that D must be finite. For otherwise, because G is locally finite, D contains a ray, say R. Therefore, there exists some $k \geq 1$ such that $R \cap H^i \neq \emptyset$ for $k \leq i \leq k+3$. Let R' denote a subpath of R between H^k and H^{k+3}. Then since all X_n^1, Y_n^1, and disjoint paths in $P_{n_{s,i}}^{s,i}$ from y_i^s and x_{i+1}^s to H^n ($n \geq 1, s \geq 2$ and $1 \leq i \leq k_s - 1$) are $(H^1, \ldots, H^n)$-forward, we see that R' is contained in some $T_{n_{s,i}}$-bridge D' of $B_{1,n_{s,i}}$ for all sufficiently large $n_{s,i}$. However, $\{x_i^r, y_i^r : k \leq r \leq k+3, 1 \leq i \leq k_r\} \subseteq V(T_{n_{s,i}})$, which shows that D' would have at least four attachments on $T_{n_{s,i}}$, a contradiction (because $T_{n_{s,i}}$ is a Tutte subgraph of $B_{1,n_{s,i}}$).

Now that D is finite and since all X_n^1, Y_n^1, and disjoint paths $P_{n_{s,i}}^{s,i}$ from y_i^s and x_{i+1}^s to H^n ($n \geq 1, s \geq 2$ and $1 \leq i \leq k_s - 1$) are $(H^1, \dots, H^n)$-forward, we see that D is a $T_{n_{s,i}}$-bridge of $B_{1,n_{s,i}}$ for all sufficiently large $n_{s,i}$. Thus D has at most three attachments on T, and if D contains an edge of H then it has just two attachments. ∎

We now state and prove the main result of this section.

Theorem 5.5 *Let G be an infinite locally finite plane graph with no dividing cycles, let C be a facial cycle of G and $e \in E(C)$, and assume that G is $(4, C)$-connected. Suppose G has at least two ends. Then G contains a C-Tutte subgraph T such that $e \in E(T)$ and the closure of T in $|G|$ is a circle.*

Proof. First, we construct a sequence $((G_i, C_i, u_i, v_i, w_i) : i = 1, 2, \cdots)$. Let $G_1 = G$ and $C_1 = C$, let u_1, v_1 be the vertices of G incident with e, and let w_1 be the neighbor of v_1 in $C - u_1$. Suppose we have constructed $(G_i, C_i, u_i, v_i, w_i)$ for some positive integer $i \geq 1$, where G_i is an infinite plane graph, C_i is a facial cycle of G_i, $u_i v_i w_i$ is a path in C_i, and G_i is $(4, C_i)$-connected.

If there is no finite cycle C_i^* in G_i such that $C_i^* \cap C_i = \emptyset$ and $C_i \subseteq I_{G_i}(C_i^*)$, then we stop this process. Otherwise, by applying Lemma 3.3 (with G_i, C_i, u_i, v_i playing the roles of G, C, u, v, respectively), there exist a plane graph G_{i+1}, a facial cycle C_{i+1} of G_{i+1}, and a path $u_{i+1} v_{i+1} w_{i+1}$ in C_{i+1} such that

(1) G_{i+1} is $(4, C_{i+1})$-connected and $G_{i+1} - v_{i+1}$ is 2-connected;

(2) $G_{i+1} - \{u_{i+1} v_{i+1}, v_{i+1} w_{i+1}\} \subseteq G_i$, and no edge of G_i joins a vertex of $G_{i+1} - V(C_{i+1})$ to a vertex of $G_i - V(G_{i+1})$;

(3) $(G_i + \{u_{i+1} v_{i+1}, v_{i+1} w_{i+1}\}) - (V(G_{i+1}) - V(C_{i+1}))$ is finite and has a plane representation in which C_i and C_{i+1} are facial cycles;

(4) $v_{i+1} \neq v_i$ and $(C_{i+1} - v_{i+1}) \cap C_i = \emptyset$;

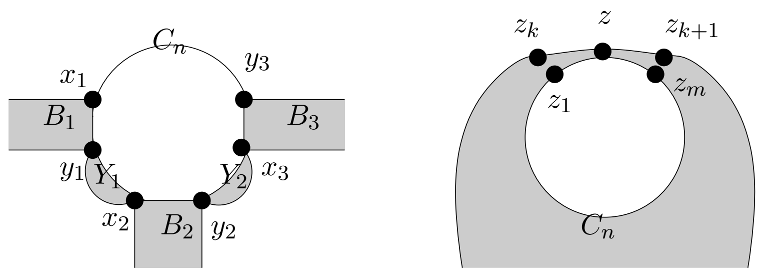
(5) for any subgraph (finite or infinite) X of G_{i+1} with $C_{i+1} \subseteq X$, and for any C_{i+1}-Tutte subgraph (finite or infinite) P_{i+1} of X containing $u_{i+1} v_{i+1}$ such that v_{i+1} has exactly one neighbor, say z_{i+1}, in $P_{i+1} - u_{i+1}$, there is a C_i-Tutte subgraph P_i of $G_i - (V(G_{i+1}) - V(X))$ containing $u_i v_i$ such that $P_{i+1} - v_{i+1} \subseteq P_i$, and $P_i - V(P_{i+1} - v_{i+1})$ is a u_{i+1}-z_{i+1} path.

By (4), $C_i \cap C_{i+2} = \emptyset$, for all $i \geq 1$. If the above construction does not stop in finitely many steps, we have $G = \bigcup_{i \geq 1} I(C_i)$, and G is 2-indivisible (i.e. has just one end), a contradiction. So the above construction must stop in finitely many steps. Let $((G_i, C_i, u_i, v_i, w_i) : i = 1, 2, \cdots, n)$ denote the maximum sequence constructed by this procedure.

By (5), it suffices to show that G_n has a C_n-Tutte subgraph P_n containing $v_n u_n$, and the closure of P_n in $|G_n|$ is a circle.

By Corollary 3.6, we work with a nice embedding of G_n in which C_n is a facial cycle. By the maximality of the sequence $((G_i, C_i, u_i, v_i, w_i) : i = 1, 2, \cdots, n)$, any finite cycle D with $C_n \subseteq I_{G_n}(D)$ must intersect C_n. So we have two cases to consider.

First, assume that for any finite cycle D in G_n for which $C_n \subseteq I_{G_n}(D)$, we must have $|V(D \cap C_n)| \geq 2$. Then there exist vertices $x_1, y_1, \dots, x_k, y_k$ (not necessarily distinct) on C_n in counter clockwise order such that for $1 \leq i \leq k$, $x_i \neq y_i$, the $\{x_i, y_i\}$-bridge of G_n containing $H_i := y_i C_n x_i$ is an infinite block (denoted by B_i), and the $\{y_i, x_{i+1}\}$-bridge of G_n containing



(a) (b)

Figure 18: The graph G_n.

$x_{i+1}C_ny_i$ is a finite graph (denoted by Y_i), where $x_{k+1} = x_1$. See Figure 18(a). (B_i and Y_i are uniquely defined since G is $(4, C)$-connected.) Note that each (B_i, H_i, x_i, y_i) is a 4-tuple. We apply Lemma 5.4 to each B_i to obtain an H_i-Tutte subgraph T_i of B_i such that (i) $v_nu_n \in E(T_i)$ whenever $v_nu_n \in E(B_i)$, and (ii) the closure of T_i in $|B_i|$ is an arc between x_i and y_i. If $|V(x_{i+1}C_ny_i)| \leq 3$ then $Y_i = x_{i+1}C_ny_i$ (since G is $(4, C)$-connected), and let $Q_i := Y_i$. If $|V(x_{i+1}C_ny_i)| \geq 4$, then $Y_i + x_{i+1}y_i$ is 2-connected; and we may apply Lemma 3.1 to find an $x_{i+1}C_ny_i$-Tutte path Q_i in $Y_i + x_{i+1}y_i$ from x_{i+1} to y_i and containing an edge of $x_{i+1}C_ny_i$ (which must be v_nu_n when $v_nu_n \in E(Y_i)$). Let $P_n := \bigcup_{i=1}^{k}(T_i \cup Q_i)$. Then P_n is a C_n-Tutte subgraph of G_n, and the closure of P_n in $|G_n|$ is a circle.

Now assume that there exists a finite cycle D in G_n such that $C_n \subseteq I_{G_n}(D)$ and $|V(D \cap C_n)| = 1$. Let $z \in V(D \cap C_n)$. Then by the maximality of n, for any finite cycle D^* in G_n for which $C_n \subseteq I_{G_n}(D^*)$, $V(D^* \cap C_n) = \{z\}$. Let $z_1, \ldots, z_m$ denote the neighbors of z in G_n which occur around z in clockwise order, with z_1, z, z_m on C_n in clockwise order. Let $1 \leq k \leq m$ be such that $z_k, z_{k+1} \notin V(I_{G_n}(D^*)) - V(D^*)$ for any finite cycle D^* with $C_n \subseteq I_{G_n}(D^*)$. See Figure 18(b). Let B denote the graph obtained from G_n by deleting z, adding two new vertices x and y, and adding the edges xz_i $(1 \leq i \leq k)$ and yz_j $(k+1 \leq j \leq m)$. Let H denote the subgraph of B induced by $(E(C_n) - \{zz_1, zz_m\}) \cup \{xz_1, yz_m\})$. Then (B, H, x, y) is a 4-tuple. By applying Lemma 5.4, there is an H-Tutte subgraph T_n of B such that $v_nu_n \in E(T_n)$ and the closure of T_n in $|B|$ is an arc between x and y. Now let P_n be obtained from T_n by identifying x and y back to z. Then P_n is a C_n-Tutte subgraph of G_n, and the closure of P_n in $|G_n|$ is a circle. ∎

We now complete the proof of Theorem 1.4. Let G be an infinite locally finite 4-connected plane graph with no dividing cycles. Let C be a facial cycle of G and $e \in E(C)$. Note that C exists since G has no dividing cycle.

If G has just one end, then Theorem 1.4 follows from Theorem 1.3. So may assume that G has at least two ends. By Theorem 5.5, G contains a C-Tutte subgraph T such that $e \in E(T)$, and the closure of T in $|G|$ is a circle. Since G is 4-connected, T is a spanning subgraph of G. So the closure of T in $|G|$ is a Hamilton circle.

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