Contractible Cliques in k-Connected Graphs*

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Abstract

Kawarabayashi proved that for any integer $k \geq 4$, every k-connected graph contains two triangles sharing an edge, or admits a k-contractible edge, or admits a k-contractible triangle. This implies Thomassen's result that every triangle-free k-connected graph contains a k-contractible edge. In this paper, we extend Kawarabayashi's technique and prove a more general result concerning k-contractible cliques.

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1 Introduction

A graph is k-connected if it has at least $k+1$ vertices and contains no vertex cut of size smaller than k. An edge (or a subgraph) in a k-connected graph is *k-contractible* if its contraction results in a k-connected graph. Tutte [6] showed that if G is a 3-connected graph then $G = K_4$ or G contains a 3-contractible edge. This result is used to show that all 3-connected graphs can be obtained from K_4 by two simple operations. Those 4-connected graphs without 4-contractible edges are characterized in [2] and [4].

Thomassen [5] showed that for $k \geq 4$, every k-connected graph contains a triangle or admits a k-contractible edge. This result is then used in [5] to prove a conjecture of Lovász. Extending techniques of Egawa [1], Kawarabayashi [3] improved Thomassen's result by showing that for $k \geq 4$, every k-connected graph contains two triangles sharing an edge, or admits a k-contractible edge not contained in any triangle, or admits a k-contractible triangle which does not share an edge with any other triangle.

A clique in a graph is a maximal complete subgraph, and a clique of size i is called an *i-clique*. (Note that if two cliques share an edge then both cliques are of size at least 3.) With this notation, Kawarabayashi's result can be stated as follows. For any integer $k \geq 4$, every k-connected graph contains two triangles sharing an edge, or admits a k-contractible i-clique for some $2 \leq i \leq 3$.

We aim to investigate the existence of a k-contractible subgraph of larger size in a k-connected graph. It turns out that the existence of such subgraphs depends on the number of triangles sharing a common edge. We are able to modify Kawarabayashi's method and prove the following more general result.

(1.1) Theorem. *Let $t \geq 0$ and $k \geq \max\{4, t+3\}$ be integers, and let G be a k-connected graph. Then one of the following holds.*

(i) There is an edge contained in $t+1$ triangles in G.

(ii) There exist two cliques in G sharing at least one edge.

(iii) There exist in G a clique of size at least 4 and a clique of size at least 3 whose intersection is non-empty.

(iv) There is a k-contractible clique in G of size at most $t+2$.

When $t = 0$ and G is triangle-free, (i), (ii), and (iii) of Theorem (1.1) cannot hold. Hence, Theorem (1.1) implies that G admits a k-contractible edge, and we obtain Thomassen's result as a consequence. When $t = 1$ and no two triangles in G share an edge, (i), (ii), and (iii) cannot hold. Hence, Kawarabayashi's result follows from Theorem (1.1).

We consider simple graphs only. Let G be a graph. We use $V(G)$ and $E(G)$ to denote the vertex set and edge set of G, respectively. For any $x \in V(G)$, $N_G(x)$ denotes the neighborhood of x in G, and we write $d_G(x) = |N_G(x)|$. Let H be a subgraph of G. Then $N_G(H)$ denotes the set of vertices of $G - V(H)$ each of which is adjacent to a vertex in $V(H)$. When H is connected, we use G/H to denote the graph obtained from G by contracting H. Also, for any $e \in E(G)$, we use G/e to denote the graph obtained from G by contracting e.

To prove Theorem (1.1), we first observe that if an i-clique in a k-connected graph is not k-contractible, then its vertex set must be contained in a vertex cut of size at most $k+i-2$ (unless G is small). We then define a collection of vertex cuts arising from non-contractible cliques. In section 2, we derive properties about those cuts and associated components. It turns out that we only need to consider those cuts of size at most $k+1$. We complete the proof of Theorem (1.1) in Section 3.

2 Cuts and components

Let G be a k-connected graph. Let K be an i-clique in G, where $i \geq 2$, and let u denote the vertex of G/K representing the contraction of K. Suppose G/K is not k-connected. Then either there is a vertex cut S' of G/K such that $|S'| \leq k-1$ or G/K is a complete graph on at most k vertices. In the former case, since G is k-connected, S' is not a cut in G, and hence, $u \in S'$. Note that $S := (S' - \{u\}) \cup V(K)$ is a cut in G. Since G is k-connected and because $|S'| \leq k-1$, we have $k \leq |S| \leq k+i-2$. Therefore, if an i-clique K in G is not k-contractible then either $V(K)$ is contained in a cut of size at least k and at most $k+i-2$ or G/K is a complete graph on at most k vertices.

Again, let G be a k-connected graph. For any clique K in G which is not k-contractible, let $\mathcal{C}_K(G)$ denote the collection of minimum cuts in G containing $V(K)$. (Note that if $\mathcal{C}_K(G) = \emptyset$ then G/K is a complete graph with at most k vertices.) Thus, if $S \in \mathcal{C}_K(G)$ and T is a cut in G containing $V(K)$ then $|T| \geq |S|$, and $T \in \mathcal{C}_K(G)$ if, and only if, $|T| = |S|$. Define $\mathcal{C}(G) = \bigcup_K \mathcal{C}_K(G)$, where the union is taken over all cliques K in G which are not k-contractible. For $i \geq 2$ and $k \leq j \leq k+i-2$, let $\mathcal{C}_i^j(G) := \{S \in \mathcal{C}(G) : |S| = j$ and $S \in \mathcal{C}_K(G)$ for some i-clique K in $G\}$. The following observation shows when $\mathcal{C}(G) \neq \emptyset$.

(2.1) Lemma. *Let $t \geq 0$ and $k \geq \max\{4, t+3\}$ be integers, and let G be a k-connected graph. Then one of the following holds.*

- *(i) There is an edge contained in $t+1$ triangles in G.*
- *(ii) There exist two cliques in G sharing at least one edge.*
- *(iii) There is a k-contractible clique in G of size at most $t+2$.*
- *(iv) $\mathcal{C}(G) \neq \emptyset$.*

Proof. Suppose (i) fails. Then every clique in G has size at most $t+2$, which implies that G is not a complete graph (because $k \geq t+3$). Let x, y be two non-adjacent vertices of G. Then any clique K in G contains x or y but not both; for otherwise, either K is k-contractible ((iii) holds), or G/K is not complete, which implies $\mathcal{C}_K(G) \neq \emptyset$, and hence, (iv) holds.

Let X be a clique in G containing x. Then $y \notin V(X)$. We may assume that G/X is a complete graph on at most k vertices; for otherwise, either X is k-contractible ((iii) holds) or $\mathcal{C}_X(G) \neq \emptyset$ ((iv) holds). So let Y denote a clique in G containing $V(G) - V(X)$. Then $V(X) \cup V(Y) = V(G)$.

We may assume $|V(X \cap Y)| \leq 1$; for otherwise (ii) holds. Hence, $|V(G)| \leq |V(X)| + |V(Y)| \leq |V(G)| + 1$. By symmetry, we may assume that $|V(X)| \leq |V(Y)|$. Then $|V(Y)| \geq 3$ (because G is k-connected and $k \geq 4$). Now x must have at least two neighbors in Y; for otherwise, $d_G(x) \leq |V(X)| \leq t+2 \leq k-1$, a contradiction. Therefore, Y shares an edge with a clique in G containing x, which implies (ii). □

Our second lemma concerns the sizes of components associated with cuts in $\mathcal{C}(G)$.

(2.2) Lemma. *Let $t \geq 0$ and $k \geq \max\{4, t+3\}$ be integers, and let G be a k-connected graph. Then one of the following holds.*

(i) There is an edge contained in $t+1$ triangles in G.

(ii) There exist two cliques in G sharing at least one edge.

(iii) There is a k-contractible clique in G of size at most $t+2$.

(iv) $\mathcal{C}(G) \neq \emptyset$, and for any $S \in \mathcal{C}(G)$ and any component H of $G-S$, we have $|V(H)| \geq k-t$, and if $k=4$ and $S \in \mathcal{C}_3^5(G)$ then $|V(H)| \geq 4$.

Proof. By Lemma (2.1), if $\mathcal{C}(G) = \emptyset$ then (i), (ii) or (iii) holds. So we may assume that $\mathcal{C}(G) \neq \emptyset$, and let $S \in \mathcal{C}(G)$. Without loss of generality, we may assume that $S \in \mathcal{C}_i^j(G)$, where $i \geq 2$ and $k \leq j \leq k+i-2$, and let K be an i-clique such that $S \in \mathcal{C}_K(G)$. Note that $|S - V(K)| \leq k-2$. Let H be a component of $G-S$.

First, assume $|V(H)| = 1$. Let x denote the only vertex in $V(H)$. Since G is k-connected, $d_G(x) \geq k$. Therefore, since $|S - V(K)| \leq k-2$, we see that x has at least two neighbors in $V(K)$. Thus, $i \geq 3$ (since K is a clique) and (ii) holds.

So assume $|V(H)| \geq 2$, and let $xy \in E(H)$. We may assume that xy is contained in at most t triangles; for otherwise we have (i). Thus $|N_G(x) \cap N_G(y)| \leq t$. We may further assume that x and y each have at most one neighbor in K, as otherwise, $i \geq 3$ (since K is a clique) and (ii) holds. Therefore, $|V(H)| \geq |N_G(x) \cup N_G(y)| - |S - V(K)| - 2 \geq |N_G(x)| + |N_G(y)| - |N_G(x) \cap N_G(y)| - (k-2) - 2 \geq 2k - t - k = k - t$. As a consequence, (iv) holds when $k \neq 4$ or $S \in \mathcal{C}_3^5(G)$.

Now let us consider the case when $k = 4$ and $S \in \mathcal{C}_3^5(G)$. Then K is a 3-clique and $|S - V(K)| = 2$. Since $k \geq t+3$ and $k = 4$, we see that $t \leq 1$. Suppose $|V(H)| < 4$. Since $|V(H)| \geq k - t \geq 3$, $|V(H)| = 3$ and $t = 1$. If any vertex of H has two neighbors in K, then we see that (i) holds (since $t = 1$). So we may assume that each vertex of H has at most one neighbor in K. Since G is 4-connected, this forces each vertex of H to be adjacent to at least one vertex in $S - V(K)$. Since $|S - V(K)| = 2$ and $|V(H)| = 3$, at least two vertices of H must share a neighbor in $S - V(K)$. If H is a triangle then (i) holds (since $t = 1$). So we may assume that H is a path. Again, since G is 4-connected, the two degree 1 vertices of H are adjacent to both vertices in $S - V(K)$, and the degree 2 vertex of H is adjacent to one vertex in $S - V(K)$. This implies (i). □

The next lemma will allow us to focus on those cuts from $\mathcal{C}(G)$ whose size is at most $k+1$.

(2.3) Lemma. *Let $t \geq 0$ and $k \geq \max\{4, t+3\}$ be integers, and let G be a k-connected graph. Then one of the following holds.*

(i) There is an edge contained in $t+1$ triangles in G.

(ii) There exist two cliques in G sharing at least one edge.

(iii) There exist in G a clique of size at least 4 and a clique of size at least 3 whose intersection is non-empty.

(iv) There is a k-contractible clique in G of size at most $t+2$.

(v) $\mathcal{C}(G) \neq \emptyset$, and for any $S \in \mathcal{C}(G)$ and for any component H of $G-S$, some edge of H belongs to a unique clique in G whose size is 2 or 3. Moreover, if an edge of H is contained in a clique in G of size at least 4 then some edge of H is not contained in any triangle.

Proof. By Lemma (2.1), we may assume $\mathcal{C}(G) \neq \emptyset$, as otherwise (i), (ii) or (iv) holds. So we may assume that $\mathcal{C}(G) \neq \emptyset$. Let $S \in \mathcal{C}(G)$. Without loss of generality, assume that L is an l-clique such that $S \in \mathcal{C}_L(G)$. Note that $k \leq |S| \leq k + l - 2$.

Let H be a component of $G - S$. We may assume that $|V(H)| \geq k - t \geq 3$; for otherwise, it follows from Lemma (2.2) that (i) or (ii) or (iv) holds. If every edge of H belongs to a unique clique in G of size at most 3, then (v) holds. If some edge of H is contained in two cliques in G then (ii) holds. So we may assume that some edge e of H is contained in a j-clique in G, say J, with $j \geq 4$. Let $|V(J \cap H)| = s$. Clearly $s \leq j$.

We may assume that no two vertices of $J \cap H$ share a common neighbor outside J, for otherwise (ii) holds. Thus $|N(J \cap H)| \geq s(k-(j-1))+(j-s)$. We may also assume that each vertex of $J \cap H$ has at most one neighbor in L; otherwise because L is a clique, (ii) holds. Hence, $|N(J \cap H) - S| \geq |N(J \cap H)| - |S| + (|L| - s) \geq s(k-(j-1))+(j-s)-(k+l-2)+(l-s) = (s-1)k-(s-1)j-s+2$. Since $k \geq t+3$ and $t+2 \geq j$, $|N(J \cap H) - S| \geq (s-1)(t+3)-(s-1)(t+2)-s+2 = 1$.

Thus, $|V(H) - V(J \cap H)| \geq |N(J \cap H) - S| \geq 1$. So at least one vertex in H does not belong to $V(J)$. Therefore, there is an edge e' of H which has exactly one incident vertex in J. If e' belongs to a triangle in G, then (iii) holds. If e' does not belong to any triangle in G, then (v) holds. □

For a k-connected graph G, let $\mathcal{C}'(G) = \mathcal{C}_2^k(G) \cup \mathcal{C}_3^k(G) \cup \mathcal{C}_3^{k+1}(G)$. Note that when (v) of Lemma (2.3) holds, some edge of H is contained in a unique clique in G of size at most 3, and so, $\mathcal{C}'(G) \neq \emptyset$ or G has a contractible clique of size at most 3.

(2.4) Lemma. *Let $t \geq 0$ and $k \geq \max\{4, t+3\}$ be integers, and let G be a k-connected graph. Then one of the following holds.*

(i) There is an edge contained in $t+1$ triangles in G.

(ii) There exist two cliques in G sharing at least one edge.

(iii) There exist in G a clique of size at least 4 and a clique of size at least 3 whose intersection is non-empty.

(iv) There is a k-contractible clique in G of size at most $t+2$.

(v) $\mathcal{C}'(G) \neq \emptyset$, and for any $S, S' \in \mathcal{C}'(G)$ and for any component H of $G - S$, $V(H) \not\subseteq S'$.

Proof. Assume that (i) – (iv) fail. Then by (v) of Lemma (2.3), $\mathcal{C}'(G) \neq \emptyset$. Let $S, S' \in \mathcal{C}'(G)$, H be a component of $G - S$, and H' be a component of $G - S'$. Let $W = G - (S \cup V(H))$ and $W' = G - (S' \cup V(H'))$. Let H_1, H_2, and H_3 denote $V(H \cap H')$, $V(H) \cap S'$, and $V(H \cap W')$, respectively. Let W_1, W_2, and W_3 denote $V(W \cap H')$, $V(W) \cap S'$, and $V(W \cap W')$, respectively. Let Q_1, Q_2, and Q_3 denote $S \cap V(H')$, $S \cap S'$, and $S \cap V(W')$, respectively. See Figure 1.

Suppose (v) fails as well, with $H_1 = \emptyset = H_3$. Then by by (v) of Lemma (2.3), $H_2 = V(H)$ contains two adjacent vertices x and y of G such that xy belongs to a unique clique in G of size at most 3. In particular, $|N_G(x) \cap N_G(y)| \leq 1$. Thus, since G is k-connected, $|N_G(x) \cup N_G(y)| \geq 2k-1$.

We claim that $|H_2| \geq k-1$. If $|S| = k$ then $|H_2| \geq |N_G(x) \cup N_G(y)| - |S| \geq 2k-1-k = k-1$. Now suppose $|S| = k+1$. Then there is a 3-clique T such that $S \in \mathcal{C}_T(G)$. Since we assume (ii) fails, T shares no edge with any other clique. Hence both x and y have at most one neighbor in

	H	S	W
H'	H_1	Q_1	W_1
S'	H_2	Q_2	W_2
W'	H_3	Q_3	W_3

Figure 1: Cuts and components

T. Therefore, $|(N_G(x) \cup N_G(y)) \cap S| \leq k$, and so, $|H_2| \geq |(N_G(x) \cup N_G(y)) - S| \geq |(N_G(x) \cup N_G(y))| - |(N_G(x) \cup N_G(y)) \cap S| \geq 2k - 1 - k = k - 1$.

Similarly, we can show that if $H_1 = W_1 = \emptyset$ then $|Q_1| \geq k - 1$, if $W_1 = W_3 = \emptyset$ then $|W_2| \geq k - 1$, and if $H_3 = W_3 = \emptyset$ then $|Q_3| \geq k - 1$. We distinguish three cases.

Case 1. $|S'| = k$ and $|S| = k$.

In this case, $|Q_2 \cup W_2| = |S'| - |H_2| \leq k - (k - 1) = 1$. Therefore, $W_1 \neq \emptyset$ or $W_3 \neq \emptyset$; as otherwise, $|W_2| = |V(W)| \geq k - 1 \geq 3$, a contradiction. So by symmetry, assume $W_1 \neq \emptyset$. Then $Q_1 \cup Q_2 \cup W_2$ is a cut in G. Since G is k-connected, $|Q_1 \cup Q_2 \cup W_2| \geq k$, and so, $|Q_1| \geq k - 1$. Thus $|Q_2 \cup Q_3| = |S| - |Q_1| \leq 1$, and hence $|Q_2 \cup Q_3 \cup W_2| \leq 2$. Therefore, since G is k-connected and $k \geq 4$, $Q_2 \cup Q_3 \cup W_2$ cannot be a cut in G. So $W_3 = \emptyset$. Since $H_3 = \emptyset$, we have $|Q_3| = |V(W')| \geq k - 1 \geq 3$, a contradiction.

Case 2. $|S'| = k + 1$ and $|S| = k$, or $|S'| = k$ and $|S| = k + 1$.

Suppose $|S'| = k+1$ and $|S| = k$. Then $|Q_2 \cup W_2| = |S'| - |H_2| \leq 2$. So $W_1 \neq \emptyset$ or $W_3 \neq \emptyset$; for otherwise, $|W_2| = |V(W)| \geq k - 1 \geq 3$, a contradiction. By symmetry, we may assume $W_1 = \emptyset$. Then $|Q_1 \cup Q_2 \cup W_2| \geq k$ because G is k-connected. Hence, $|Q_1| \geq k - 2$. Since $|S| = k$, we have $|Q_2 \cup Q_3| \leq 2$. Then $W_3 \neq \emptyset$, as otherwise, $|Q_3| \geq k - 1 \geq 3$, a contradiction. So $Q_3 \cup Q_2 \cup W_2$ is a cut in G. Since G is k-connected and $k \geq 4$, we must have $k = 4$, $|Q_2| = 0$, and $|Q_3| = |W_2| = 2$. Since $|S'| = 5$, we have $|V(H)| = |H_2| = 3$ and $S' \in \mathcal{C}_3^5(G)$. Hence H is a triangle, and by (v) of Lemma (2.3), no two vertices of H has a common neighbor in S. However this would force some vertex in $V(H)$ to have degree at most 3 in G, a contradiction.

Now assume $|S'| = k$ and $|S| = k + 1$. Then $|Q_2 \cup W_2| = |S'| - |H_2| \leq 1$. Therefore, $W_1 \neq \emptyset$ or $W_3 \neq \emptyset$, for otherwise, $|W_2| = |V(W)| \geq k - 1 \geq 3$, a contradiction. Let T denote a 3-clique such that $S \in \mathcal{C}_T(G)$. By symmetry, assume that $V(T) \subseteq Q_1 \cup Q_2$.

Suppose $W_3 = \emptyset$. Then $W_1 \neq \emptyset$ and $Q_1 \cup Q_2 \cup W_2$ is a cut in G containing $V(T)$. Since $S \in \mathcal{C}_T(G)$, $|Q_1 \cup Q_2 \cup W_2| \geq |S| = k + 1$. This, together with $|Q_2 \cup W_2| \leq 1$, implies $|Q_1| \geq k$, and hence, $|Q_2 \cup Q_3| = |S| - |Q_1| \leq 1$. On the other hand, since $H_3 = \emptyset = W_3$, $|Q_3| = |V(W')| \geq k - 1 \geq 3$, a contradiction.

So $W_3 \neq \emptyset$. Then $W_2 \cup Q_2 \cup Q_3$ is a cut in G, and hence, $|W_2 \cup Q_2 \cup Q_3| \geq k$. Thus, since $|Q_2 \cup W_2| \leq 1$, $|Q_3| \geq k - 1$, and so, $|Q_1 \cup Q_2| = |S| - |Q_3| \leq 2$. Now $|Q_1 \cup Q_2 \cup W_2| \leq 3$, which implies that $Q_1 \cup Q_2 \cup W_2$ cannot be a cut in G. So $W_1 = \emptyset$. Since $H_1 = \emptyset$, $|Q_1| = |V(H')| \geq k - 1 \geq 3$, a contradiction.

Case 3. $|S| = |S'| = k + 1$.

Then $|W_2 \cup Q_2| = |S'| - |H_2| \leq k+1-(k-1) = 2$. Note that $W_1 \neq \emptyset$ or $W_3 \neq \emptyset$, for otherwise, $|W_2| = |V(W)| \geq k - 1 \geq 3$, a contradiction. Let T denote a 3-clique such that $S \in \mathcal{C}_T(G)$. By symmetry, assume $V(T) \subseteq Q_1 \cup Q_2$.

First, assume $W_1 = \emptyset$. Then $|Q_1| = |V(H')| \geq k-1$ (since $H_1 = \emptyset$) and $W_3 \neq \emptyset$. Now $W_3 \neq \emptyset$ implies that $W_2 \cup Q_2 \cup Q_3$ is a cut in G, and hence, $|W_2 \cup Q_2 \cup Q_3| \geq k \geq 4$. Also $|Q_1| \geq k - 1$ implies $|Q_2 \cup Q_3| = |S| - |Q_1| \leq k + 1 - (k - 1) = 2$. Since $|W_2 \cup Q_2| \leq 2$ and $|Q_2 \cup Q_3| \leq 2$, we have $k = 4$, $Q_2 = \emptyset$, $|Q_3| = |W_2| = 2$, $|S'| = |S| = 5$, and $|H_2| = |V(H)| = 5 - 2 = 3$. Thus, $S \in \mathcal{C}_3^5(G)$ and $|V(H)| = 3$, contradicting (iv) of Lemma (2.2) (since we assume (i), (ii), (iii) of Lemma (2.2) fail).

Now assume $W_1 \neq \emptyset$. Then $Q_1 \cup Q_2 \cup W_2$ is a cut in G containing $V(T)$. Since $S \in \mathcal{C}_T(G)$, we see that $|Q_1 \cup Q_2 \cup W_2| \geq |S| = k + 1$. Since $|W_2 \cup Q_2| \leq 2$, $|Q_1| \geq k - 1$, and so, $|Q_2 \cup Q_3| \leq |S| - |Q_1| \leq (k + 1) - (k - 1) = 2$. If $W_3 = \emptyset$ then $|Q_3| = |V(W')| \geq k - 1 \geq 3$ (since $H_3 = \emptyset$), a contradiction. So $W_3 \neq \emptyset$. Thus $W_2 \cup Q_2 \cup Q_3$ is a cut in G, and hence, $|W_2 \cup Q_2 \cup Q_3| \geq k \geq 4$. This implies that $k = 4$, $Q_2 = \emptyset$, $|W_2| = 2$, $|Q_3| = 2$, $|S| = |S'| = 5$, and $|H_2| = |V(H)| = 5 - 2 = 3$. Again, $S \in \mathcal{C}_3^5(G)$ and $|V(H)| = 3$, which contradicts (iv) of Lemma (2.2). □

3 Proof of the main result

In this section, we prove Theorem (1.1). Our argument is similar to that in [3] which was first introduced by Egawa [1]. Let G be a k-connected graph and let $t \geq 0$ be an integer, and assume $k \geq \max\{4, t+3\}$. We first show that it suffices to consider cuts in $\mathcal{C}'(G)$. We then complete the proof by investigating the sizes of components associated with cuts in $\mathcal{C}'(G)$.

Suppose for a contradiction that Theorem (1.1) is false. Then we have the following.

(1) No edge of G is contained in $t + 1$ triangles.

(2) No two cliques in G share an edge.

(3) No clique in G of size at least 4 shares a vertex with a clique in G of size at least 3.

(4) No clique in G is k-contractible.

Therefore, it follows from (v) of Lemma (2.3) that $\mathcal{C}'(G) \neq \emptyset$. We choose $S \in \mathcal{C}'(G)$ and a component H of $G - S$ such that

(5) $|V(H)|$ is minimum.

By (1) – (4) and by (iv) of Lemma (2.2), $|V(H)| \geq k - t \geq 3$. Let $W = G - (S \cup V(H))$. Next we show that

(6) $S \in \mathcal{C}_3^{k+1}(G)$.

Suppose $S \notin \mathcal{C}_3^{k+1}(G)$. Then $|S| = k$. By (1) – (4) and (v) of Lemma (2.3), we may choose an edge of H which belongs to a unique clique K in G of size at most 3. Let $S' \in \mathcal{C}_K(G)$. Then $|S'| \leq k + 1$. Let H' be a component of $G - S'$ and let $W' = G - (S' \cup V(H'))$.

Let H_1, H_2, and H_3 denote $V(H \cap H')$, $V(H) \cap S'$, and $V(H \cap W')$, respectively. Let W_1, W_2, and W_3 denote $V(W \cap H')$, $V(W) \cap S'$, and $V(W \cap W')$, respectively. Let Q_1, Q_2, and Q_3 denote $S \cap V(H')$, $S \cap S'$, and $S \cap V(W')$, respectively. (See Figure 1.)

By (1) – (4) and by (v) Lemma (2.4), we have $H_1 \neq \emptyset \neq W_3$ or $H_3 \neq \emptyset \neq W_1$. By symmetry, we may assume that $H_1 \neq \emptyset \neq W_3$. Then $H_2 \cup Q_2 \cup Q_1$ is a cut in G containing $V(K)$. Therefore, since $S' \in \mathcal{C}_K(G)$ and by (5), $|H_2 \cup Q_2 \cup Q_1| \geq |S'|+1$. Since $W_3 \neq \emptyset$, $W_2 \cup Q_2 \cup Q_3$ is a cut in G, and so, $|W_2 \cup Q_2 \cup Q_3| \geq k = |S|$. This implies that $|S|+|S'| = |H_2 \cup Q_2 \cup Q_1| + |W_2 \cup Q_2 \cup Q_3| \geq (|S'|+1)+|S|$, a contradiction.

By (6), let T be a 3-clique in G such that $S \in \mathcal{C}_T(G)$.

(7) For any clique K in G containing an edge of H and for any $S' \in \mathcal{C}_K(G)$, $|S'| = k+1$.

Let K denote a clique containing an edge of H, and let $S' \in \mathcal{C}_K(G)$. Let H' be a component of $G - S'$ and let $W' = G - (S' \cup V(H'))$. Let H_1, H_2, and H_3 denote $V(H \cap H')$, $V(H) \cap S'$, and $V(H \cap W')$, respectively. Let W_1, W_2, and W_3 denote $V(W \cap H')$, $V(W) \cap S'$, and $V(W \cap W')$, respectively. Let Q_1, Q_2, and Q_3 denote $S \cap V(H')$, $S \cap S'$, and $S \cap V(W')$, respectively. (See Figure 1.) Note that $V(K) \subseteq Q_2 \cup H_2$.

Suppose $|S'| = k$. Then $S' \in \mathcal{C}'(G)$. Hence, by (1) – (4) and by (v) of Lemma (2.4), we may assume from symmetry that $H_1 \neq \emptyset \neq W_3$. Then $H_2 \cup Q_2 \cup Q_1$ and $W_2 \cup Q_2 \cup Q_3$ are cuts in G. Since $V(K) \subseteq Q_2 \cup H_2$, it follows from the choice of S (see (5)) that $|H_2 \cup Q_2 \cup Q_1| \geq |S|+1$. Since $W_2 \cup Q_2 \cup Q_3$ is a cut in G, $|W_2 \cup Q_2 \cup Q_3| \geq k = |S'|$. This implies that $|S|+|S'| = |H_2 \cup Q_2 \cup Q_1| + |W_2 \cup Q_2 \cup Q_3| \geq (|S|+1)+|S'|$, a contradiction.

Thus, for any clique K containing an edge of H, if $S' \in \mathcal{C}_K(G)$ then $|S'| \geq k+1$. Hence by (v) of Lemma (2.3), every edge of H is contained in a unique clique in G which is of size 3. So $|S'| = k+1$. So we have (7).

Next, we take a spanning tree P of H, and label the edges of P as $e_1, \ldots, e_m$ such that for each $1 \leq i \leq m$ the subgraph of H induced by $\{e_1, \ldots, e_i\}$ is connected. For each $1 \leq i \leq m$, it follows from (7) that e_i belongs to a 3-clique T_i in G, and for any $S_i \in \mathcal{C}_{T_i}(G)$, we have $|S_i| = k+1$.

Let H^i be a component in $G - S_i$ and let $W^i = G - (S_i \cup V(H^i))$. Let H^i_1, H^i_2 and H^i_3 denote $V(H \cap H^i)$, $V(H) \cap S_i$ and $V(H \cap W^i)$, respectively. Let W^i_1, W^i_2 and W^i_3 denote $V(W \cap H^i)$, $V(W) \cap S_i$ and $V(W \cap W^i)$, respectively. Let Q^i_1, Q^i_2 and Q^i_3 denote $S \cap V(H^i)$, $S \cap S_i$ and $S \cap V(W^i)$, respectively. Since T is fixed, we may assume that the notation is chosen so that $V(T) \subseteq Q^i_1 \cup Q^i_2$ for all $1 \leq i \leq m$.

Note that $V(T_i) \subseteq H^i_2 \cup Q^i_2$, and $|V(T_i \cap T)| \leq 1$ (since $|V(T_i \cap H)| \geq 2$).

(8) $H^i_3 = \emptyset$, $|H^i_2| = |Q^i_3|+1$ and $|Q^i_3| \geq 1$, and $|V(H)| \geq k$.

First, assume $H^i_3 \neq \emptyset$. Then $H^i_2 \cup Q^i_2 \cup Q^i_3$ is a cut in G containing $V(K)$. Hence by (5) and since $S_i \in \mathcal{C}_K(G)$, $|H^i_2 \cup Q^i_2 \cup Q^i_3| \geq |S_i|+1 = k+2$. So $|H^i_2| \geq (k+2) - |Q^i_2 \cup Q^i_3| = |Q^i_1|+1$. If $W^i_1 \neq \emptyset$, then $Q^i_1 \cup Q^i_2 \cup W^i_2$ is a cut in G containing $V(T)$. Since $S \in \mathcal{C}_T(G)$, $|Q^i_1 \cup Q^i_2 \cup W^i_2| \geq |S| = k+1$. This shows $2k+2 = |S|+|S_i| = |H^i_2 \cup Q^i_2 \cup Q^i_3| + |Q^i_1 \cup Q^i_2 \cup W^i_2| \geq 2k+3$, a contradiction. So $W^i_1 = \emptyset$. Therefore, $|V(H)| = |H^i_1|+|H^i_2|+|H^i_3| \geq |H^i_1|+|Q^i_1|+1 = |V(H^i)|+1$. This shows that S_i and H^i contradict the choices of S and H (see (5)).

Thus, $H^i_3 = \emptyset$. It follows from (1) – (4) and (v) of Lemma (2.4) that $H^i_1 \neq \emptyset \neq W^i_3$. So $H^i_2 \cup Q^i_2 \cup Q^i_1$ is a cut in G containing $V(T_i) \cup V(T)$, and $W^i_2 \cup Q^i_2 \cup Q^i_3$ is a cut in G. Since $S \in \mathcal{C}_T(G)$ and by (5), $|H^i_2 \cup Q^i_2 \cup Q^i_1| \geq k+2$. Since G is k-connected, $|W^i_2 \cup Q^i_2 \cup Q^i_3| \geq k$. This shows that $|S|+|S_i| \geq 2k+2$. On the other hand, $|S| = |S_i| = k+1$. Hence $|H^i_2 \cup Q^i_2 \cup Q^i_1| = k+2$ and $|W^i_2 \cup Q^i_2 \cup Q^i_3| = k$. So we have $|H^i_2| = |Q^i_3|+1$. Since $|H^i_2| \geq 2$, we have $|Q^i_3| \geq 1$.

To prove $|V(H)| \geq k$, we first show that there is an edge of G whose incident vertices are contained in H^i_1. For otherwise, any $z \in V(H^i_1)$ has all its neighbors contained in $H^i_2 \cup Q^i_2 \cup Q^i_1$. Since z is adjacent to at most one vertex of T as well as to at most one vertex of T_i, and since

$|V(T_i \cap T)| \leq 1$, we see that $d_G(z) \leq |H_2^i \cup Q_1^i \cup Q_2^i| - 3 = (k+2) - 3 = k - 1$, a contradiction (since G is k-connected).

So let xy be an edge of G such that $x, y \in V(H_1^i)$. Note that x and y each have at most one neighbor in T as well as at most one neighbor in T_i. So $|(N_G(x) \cup N_G(y)) \cap (H_2^i \cup Q_2^i \cup Q_1^i)| \leq k + 1$. Also since xy is contained in only one triangle (by (2) and (7)), we see that $|N_G(x) \cap N_G(y)| \leq 1$. Therefore, since G is k-connected, $|N_G(x) \cup N_G(y)| \geq 2k - 1$. Hence $|H_1^i| \geq |N_G(x) \cup N_G(y)| - |(N_G(x) \cup N_G(y)) \cap (H_2^i \cup Q_2^i \cup Q_1^i)| \geq (2k-1) - (k+1) = k - 2$. This means $|V(H)| \geq |H_1^i| + |H_2^i| \geq (k-2) + 2 = k$, completing the proof of (8).

Next we show that

(9) $|N_G(U) \cap V(H)| \geq |U| + 1$ for all non-empty subsets U of $S - V(T)$.

Suppose for some non-empty subset U of $S - V(T)$ we have $|N_G(U) \cap V(H)| \leq |U|$. Note that $|U| \leq |S - V(T)| \leq k-2 < |V(H)|$. So $V(H) - N_G(U) \neq \emptyset$. Thus, $S^* := (S - U) \cup (N_G(U) \cap V(H))$ is a cut in G containing $V(T)$. Since $S \in \mathcal{C}_T(G)$ and $|S^*| \leq |S|$, we see $S^* \in \mathcal{C}_T(G)$. Note that $H - N_G(U)$ contains a component H^* of $G - S^*$ and $|V(H) - N_G(U)| < |V(H)|$. So S^* and H^* contradict the choices of S and H (see (5)).

(10) $N_G(Q_3^i) \cap V(H) = H_2^i$.

By (8), we have $H_3^i = \emptyset$. So $N_G(Q_3^i) \cap V(H) \subseteq H_2^i$. Since $|H_2^i| = |Q_3^i| + 1$ (by (8)) and $|N_G(Q_3^i) \cap V(H)| \geq |Q_3^i| + 1 = |H_2^i|$ (by (9)), we have (10).

(11) For any $1 \leq j \leq m$, $|\bigcup_{i=1}^{j}(N_G(Q_3^i) \cap V(H))| \leq |\bigcup_{i=1}^{j} Q_3^i| + 1$.

We prove (11) by induction on j. When $j = 1$, (11) follows from (8) and (10). So assume $j \geq 2$. If $Q_3^j \subseteq \bigcup_{i=1}^{j-1} Q_3^i$, the result follows from the induction hypothesis. Hence, we may assume $Q_3^j \not\subseteq \bigcup_{i=1}^{j-1} Q_3^i$. For convenience, let $R := Q_3^j \cap (\bigcup_{i=1}^{j-1} Q_3^i)$ and $A := (N_G(Q_3^j) \cap V(H)) \cap (\bigcup_{i=1}^{j-1} N_G(Q_3^i) \cap V(H))$. Note that $|A| = |(N_G(Q_3^j) \cap (\bigcup_{i=1}^{j-1} N_G(Q_3^i))) \cap V(H)| \geq |N_G(R) \cap V(H)|$.

We claim that $|A| \geq |R| + 1$. If $R \neq \emptyset$, then $|A| \geq |N_G(R) \cap V(H)| \geq |R| + 1$ (by (9)). Now assume $R = \emptyset$. Since $\{e_1, \ldots, e_j\}$ induces a connected subgraph of H, $|H_2^j \cap (\bigcup_{i=1}^{j-1} H_2^i)| \geq 1$. By (10), $A = (N_G(Q_3^j) \cap (\bigcup_{i=1}^{j-1} N_G(Q_3^i))) \cap V(H) = H_2^j \cap (\bigcup_{i=1}^{j-1} H_2^i)$, and so, $|A| \geq 1$. Therefore, $|A| \geq |R| + 1$.

Thus $|\bigcup_{i=1}^{j}(N(Q_3^i) \cap V(H))| = |(N_G(Q_3^j) \cap V(H)) \cup (\bigcup_{i=1}^{j-1}(N_G(Q_3^i) \cap V(H)))| \leq (|\bigcup_{i=1}^{j-1} Q_3^i| + 1) + (|Q_3^j| + 1) - (|R| + 1) \leq |\bigcup_{i=1}^{j} Q_3^i| + 1$. This proves (11).

Since P is a spanning tree of H and $E(P) = \{e_1, \ldots, e_m\}$, we see that $\bigcup_{i=1}^{m} H_2^i = V(H)$. Note that $\bigcup_{i=1}^{m}(N_G(Q_3^i) \cap V(H)) = \bigcup_{i=1}^{m} H_2^i$. Hence it follows from (11) that $|V(H)| \leq |\bigcup_{i=1}^{m} N_G(Q_3^i)| + 1 \leq |S - V(T)| + 1 \leq k + 1 - 3 + 1 = k - 1$, contradicting (8). This completes the proof of Theorem (1.1).

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