A note on balanced bipartitions

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Abstract

A *balanced bipartition* of a graph G is a bipartition V_1 and V_2 of $V(G)$ such that $-1 \leq |V_1| - |V_2| \leq 1$. Bollobás and Scott conjectured that if G is a graph with m edges and minimum degree at least 2 then G admits a balanced bipartition V_1, V_2 such that $\max\{e(V_1), e(V_2)\} \leq m/3$, where $e(V_i)$ denotes the number of edges of G with both ends in V_i. In this note, we prove this conjecture for graphs with average degree at least 6 or with minimum degree at least 5. Moreover, we show that if G is a graph with m edges and n vertices, and if the maximum degree $\Delta(G) = o(n)$ or the minimum degree $\delta(G) \to \infty$, then G admits a balanced bipartition V_1, V_2 such that $\max\{e(V_1), e(V_2)\} \leq (1 + o(1))m/4$, answering a question of Bollobás and Scott in the affirmative. We also provide a sharp lower bound on $\max\{e(V_1, V_2) : V_1, V_2 \text{ is a balanced bipartition of } G\}$, in terms of size of a maximum matching, where $e(V_1, V_2)$ denotes the number of edges between V_1 and V_2.

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1 Introduction

A classical example of bipartition problems is the *Maximum Bipartite Subgraph Problem* which asks for a partition of the vertex set of a graph into V_1, V_2 maximizing $e(V_1, V_2)$, the number of edges between V_1 and V_2. Edwards [7, 8] proved that every graph with m edges admits a bipartition V_1, V_2 such that $e(V_1, V_2) \geq \frac{m}{2} + \frac{1}{4}\sqrt{2m + \frac{1}{4}} - \frac{1}{8}$. This bound is best possible for the complete graphs K_{2n+1}. For special classes of graphs, the main term in this bound may be improved. For example, Thomassen [12] improved the bound to $\frac{29}{26}m - \frac{7}{6}$ for 3-connected, cubic, triangle-free, planar graphs. Generalization of Edward's result to k-partitions are given in [4] (also see [3]) by Bollobás and Scott.

Judicious partition problems ask for a partition of the vertex set of a graph G so that several quantities are optimized simultaneously. The *Bottleneck Bipartition Problem* introduced by Entringer (see [11]) is such an example: Given a graph G, find a partition V_1, V_2 of $V(G)$ that minimizes $\max\{e(V_1), e(V_2)\}$, where $e(V_i)$ denotes the number of edges of G with both ends in V_i. Székely and Shahrokhi [11] showed that this problem is NP-hard. Porter [9] proved that for any graph G with m edges there is a partition V_1, V_2 of $V(G)$ such that $\max\{e(V_1), e(V_2)\} \leq m/4 + O(\sqrt{m})$, establishing a conjecture of Erdös. (A matrix version of Erdös' conjecture was raised by Entringer and was solved by Porter and Székely [10].)

The Bottleneck Bipartition Problem was also studied by Bollobás and Scott [1], who prove in [2] that for any graph G with m edges there is a bipartition V_1, V_2 of $V(G)$ such that $e(V_1, V_2) \geq \frac{m}{2} + \frac{1}{4}\sqrt{2m + \frac{1}{4}} - \frac{1}{8}$ and $\max\{e(V_1), e(V_2)\} \leq \frac{m}{4} + \frac{1}{8}\sqrt{2m + \frac{1}{4}} - \frac{1}{16}$. This result has recently been extended to k-partitions by Xu and Yu [13].

In this paper, we study the Bottleneck Bipartition Problem with the additional requirement that the bipartition V_1, V_2 be *balanced*, that is, $-1 \leq |V_1| - |V_2| \leq 1$. In general, a k-partition $V_1, \ldots, V_k$ of $V(G)$ is said to be *balanced* if $-1 \leq |V_i| - |V_j| \leq 1$ for $1 \leq i, j \leq k$. (The classical Min k-Section Problem asks for a balanced k-partition of a weighted graph that minimizes the total weight of edges joining different sets.)

A matching M of G is called a *symmetric matching* if for each edge $uv \in M$, u and v have the same degree in G. The main result of this paper is the following, where $\Delta(G)$ and $\delta(G)$ denote the maximum and minimum degree of the graph G, respectively.

Theorem 1.1 *Let G be a graph with n vertices and m edges, and let M be a symmetric matching in G of maximum cardinality. Then, G admits a balanced bipartition V_1, V_2 such that $e(V_1, V_2) \geq (m + |M|)/2$ and*

(1) $\max\{e(V_1), e(V_2)\} \leq \lfloor (m - |M| + \Delta(G) - \delta(G))/4 \rfloor$ *if n is even;*

(2) $\max\{e(V_1), e(V_2)\} \leq \lfloor (m - |M| + \Delta(G))/4 \rfloor$ *if n is odd.*

This result is best possible for $K_{1,n-1}$, $n \geq 2$. Since a maximum matching can be found in polynomial time [6], a maximum symmetric matching can also be found in polynomial time (by considering the subgraph induced by the edges whose ends have the same degree). Our proof of Theorem 1.1 implies a polynomial time algorithm for finding such a partition in any given graph.

This work was also motivated by the following problem of Bollobás and Scott [3]: Given a graph G, find a balanced partition $V_1, \ldots, V_k$ of $V(G)$ that minimizes $\max\{e(V_1), \ldots, e(V_k)\}$. In particular, Bollobás and Scott [3] made the following

Conjecture 1.2 *Every graph with m edges and minimum degree at least 2 admits a balanced bipartition V_1, V_2 such that* $\max\{e(V_1), e(V_2)\} \leq m/3$.

The complete graph K_3 shows that the bound $m/3$ is sharp. Indeed, the graphs obtained from disjoint triangles by identifying one vertex from each triangle also show that $m/3$ is sharp. The star $K_{1,n}$ shows that the requirement on minimum degree is necessary.

Bollobás and Scott [5] proved Conjecture 1.2 for regular graphs. In fact, they proved that almost every regular graph with m edges admits a balanced bipartition V_1, V_2 such that $\max\{e(V_1), e(V_2)\} < m/4$. In [15], Yan and Xu proved that every graph G with m edges and $\Delta(G) - \delta(G) \leq 1$ admits a balanced bipartition V_1, V_2 such that $\max\{e(V_1), e(V_2)\} < m/4 + O(n)$. Xu, Yan and Yu [14] proved Conjecture 1.2 for graphs G with $\Delta(G) \leq \frac{7}{5}\delta(G)$.

We prove Theorem 1.1 in Section 2. Since $\Delta(G) \leq m/12$ when $m \geq 3n$, Theorem 1.1 implies Conjecture 1.2 for graphs with average degree at least 6. We will see in Section 3 that Theorem 1.1 also implies Conjecture 1.2 for graphs with minimum degree at least 5.

Note that by taking a random balanced bipartition V_1, V_2 one expects to have $e(V_i) \leq m/4$. Bollobás and Scott [3] asked for conditions that guarantee a balanced bipartition V_1, V_2 in a graph with m edges such that $\max\{e(V_1), e(V_2)\} \leq (1 + o(1))m/4$, and suggested $\Delta(G) = o(n)$ or $\delta(G) \to \infty$ as $n \to \infty$. As a consequence of Theorem 1.1, we confirm this in Section 3.

In [3], Bollobás and Scott ask the following question: For a graph G of order n and size m, what are the largest and smallest cuts that we can guarantee with balanced bipartitions? In Section 3, we show that if G is a graph with m edges and M is a maximum matching in G, then G has a balanced bipartition V_1, V_2 such that $e(V_1, V_2) \geq m/2 + |M|/2$. This lower bound is sharp for complete graphs.

2 Balanced judicious bipartitions

In this section, we prove Theorem 1.1. For convenience, let $\Delta := \Delta(G)$, $\delta := \delta(G)$, and let $r := |M|$.

For $i \in \{\delta, \delta+1, \ldots, \Delta\}$, let $U_i = \{u : d(u) = i\}$, and let $n_i = |U_i|$, i.e., U_i and n_i are the set and the number of vertices of degree i, respectively. Let $A = \{i : \delta \leq i \leq \Delta, n_i \text{ is odd}\}$.

Since $n = \sum_{i=\delta}^{\Delta} n_i$, $|A|$ has the same parity as n. So we have two cases according the parity of n.

Case 1. n and $|A|$ are even.

Let $A = \{a_1, a_2, \ldots, a_{2t}\}$, where $\delta \leq a_1 < a_2 < \cdots < a_{2t} \leq \Delta$. Since $|U_{a_i}| = n_{a_i}$ is odd for each $i \in \{1, 2, \ldots, 2t\}$, U_{a_i} is not empty. Thus we may choose a vertex $x_i \in U_{a_i}$ that is not covered by M, and set $U'_{a_i} := U_{a_i} \setminus \{x_i\}$. Then $|U'_{a_i}|$ is even.

Clearly there exists a balanced bipartition V_1, V_2 of $V(G)$ such that

(*i*) if n_i is even, then $|U_i \cap V_1| = |U_i \cap V_2|$,

(*ii*) if n_i is odd, then $|U'_i \cap V_1| = |U'_i \cap V_2|$,

(*iii*) for each edge $uv \in M$, $\{u, v\} \cap V_j \neq \emptyset$ for $j = 1, 2$, and

(*iv*) for $i \in \{1, \ldots, t\}$ and $j = 1, 2$, $\{x_{2i-1}, x_{2i}\} \cap V_j \neq \emptyset$.

We may choose V_1, V_2 so that

(v) subject to (i)-(iv), $e(V_1, V_2)$ is maximum.

Without loss of generality, we assume that $e(V_1) \geq e(V_2)$. By the choice of V_1 and V_2, we may label the vertices of V_1 as $u_1, u_2, \ldots, u_t, u_{t+1}, \ldots, u_k$, and label the vertices of V_2 as $v_1, v_2, \ldots, v_t, v_{t+1}, \ldots, v_k$, such that $\{u_i, v_i\} = \{x_{2i-1}, x_{2i}\}$ for $i \in \{1, 2, \ldots, t\}$, and $u_i v_i \in M$ for $i \in \{k-r+1, \ldots, k\}$. Then $d(u_i) = d(v_i)$ for $t+1 \leq i \leq k$.

Since $\sum_{u \in V_1} d(u) = 2e(V_1) + e(V_1, V_2)$ and $\sum_{v \in V_2} d(v) = 2e(V_2) + e(V_1, V_2)$,

$$\begin{aligned} 2(e(V_1) - e(V_2)) &= \sum_{u \in V_1} d(u) - \sum_{v \in V_2} d(v) \\ &\leq \sum_{i=1}^{t} |d(u_i) - d(v_i)| \\ &= \sum_{i=1}^{t} (a_{2i} - a_{2i-1}) \\ &\leq \Delta - \delta. \end{aligned}$$

Therefore, $e(V_1) - e(V_2) \leq (\Delta - \delta)/2$.

We claim that for $i = 1, 2, \ldots, k-r$,

$$|N(u_i) \cap V_2| + |N(v_i) \cap V_1| \geq |N(u_i) \cap V_1| + |N(v_i) \cap V_2|,$$

and for $i = k-r+1, \ldots, k$,

$$|N(u_i) \cap V_2| + |N(v_i) \cap V_1| \geq |N(u_i) \cap V_1| + |N(v_i) \cap V_2| + 2.$$

To see this, fix $i \in \{1, \ldots, k\}$, and set $V_1' = (V_1 \setminus \{u_i\}) \cup \{v_i\}$ and $V_2' = (V_2 \setminus \{v_i\}) \cup \{u_i\}$. By the choice of V_1, V_2, we have $e(V_1', V_2') \leq e(V_1, V_2)$. Let $t := |N(u_i) \cap V_2| + |N(v_i) \cap V_1| - (|N(u_i) \cap V_1| + |N(v_i) \cap V_2|)$. If $u_i v_i \in E(G)$ then $e(V_1, V_2) \geq e(V_1', V_2') = e(V_1, V_2) - t + 2$; and otherwise, $e(V_1, V_2) \geq e(V_1', V_2') = e(V_1, V_2) - t$. Hence $t \geq 2$ if $u_i v_i \in E(G)$, and $t \geq 0$ otherwise; so the claim follows.

Note that $\sum_{i=1}^{k} |N(u_i) \cap V_1| = 2e(V_1)$, $\sum_{i=1}^{k} |N(v_i) \cap V_2| = 2e(V_2)$, and

$$e(V_1, V_2) = \sum_{i=1}^{k} |N(u_i) \cap V_2| = \sum_{i=1}^{k} |N(v_i) \cap V_1|.$$

Hence by the above claim,

$$\begin{aligned} e(V_1, V_2) &= \frac{1}{2} \sum_{i=1}^{k} (|N(u_i) \cap V_2| + |N(v_i) \cap V_1|) \\ &\geq \frac{1}{2} \left(\sum_{i=1}^{k} (|N(u_i) \cap V_1| + |N(v_i) \cap V_2|) \right) + r \\ &= e(V_1) + e(V_2) + r. \end{aligned}$$

Therefore, since $0 \leq e(V_1) - e(V_2) \leq (\Delta - \delta)/2$ and $e(V_1) + e(V_2) + e(V_1, V_2) = m$, we have $\max\{e(V_1), e(V_2)\} \leq (m - r + \Delta - \delta)/4$ and $e(V_1, V_2) \geq (m+r)/2$. So (1) of Theorem 1.1 holds.

Case 2. n and $|A|$ are odd.

Let $A = \{b_0, b_1, \ldots, b_{2s}\}$, where $\delta \le b_0 < b_1 < \cdots < b_{2s} \le \Delta$. Since $|U_{b_i}| = n_{b_i}$ is odd for each $i \in \{0, 1, 2, \ldots, 2s\}$ (possibly $s = 0$), we may choose a vertex $y_i \in U_{b_i}$ that is not covered by M. Set $U'_{b_i} := U_{b_i} \setminus \{y_i\}$. Then, $|U'_{b_i}|$ is even.

As in Case 1, there exists a balanced bipartition V_1, V_2 of $V(G)$ such that

(i) if n_i is even, then $|U_i \cap V_1| = |U_i \cap V_2|$,

(ii) if n_i is odd, then $|U'_i \cap V_1| = |U'_i \cap V_2|$,

(iii) for each edge $uv \in M$, $\{u, v\} \cap V_j \neq \emptyset$ for $j = 1, 2$, and

(iv) for $i \in \{1, \ldots, s\}$ and $j = 1, 2$, $\{y_{2i-1}, y_{2i}\} \cap V_j \neq \emptyset$.

Again, we choose V_1, V_2 so that

(v) subject to (i)-(iv), $e(V_1, V_2)$ is maximum.

Without loss of generality, we assume that $e(V_1) \ge e(V_2)$. Assume $y_0 \in V_p$ for some $p \in \{1, 2\}$. By (i)-(iv), $|V_p| = |V_{3-p}| + 1$. For $i \in \{1, 2\}$, let $V'_i = V_i$ if $i \neq p$, and otherwise let $V'_i = V_i \backslash \{y_0\}$. Then we may label the vertices of V'_1 as $u_1, u_2, \ldots, u_s, u_{s+1}, \ldots, u_k$, and label the vertices of V'_2 as $v_1, v_2, \ldots, v_s, v_{s+1}, \ldots, v_k$, such that $\{u_i, v_i\} = \{y_{2i-1}, y_{2i}\}$ for $i \in \{1, 2, \ldots, s\}$, and $u_i v_i \in M$ for $i \in \{k - r + 1, \ldots, k\}$. Then $d(u_i) = d(v_i)$ for $s + 1 \le i \le k$.

So by (i)-(iv),

$$\begin{aligned} 2(e(V_1) - e(V_2)) &= \sum_{u \in V_1} d(u) - \sum_{v \in V_2} d(v) \\ &\le d(y_0) + \sum_{i=1}^{s} |d(u_i) - d(v_i)| \\ &= b_0 + \sum_{i=1}^{s} (b_{2i} - b_{2i-1}). \end{aligned}$$

If $s = 0$ then, since $b_0 \le \Delta$, we have $e(V_1) - e(V_2) \le \Delta/2$. If $s \ge 1$ then $2(e(V_1) - e(V_2)) \le b_0 + (\Delta - b_1) < \Delta$, and hence $e(V_1) - e(V_2) < \Delta/2$. So we have $0 \le e(V_1) - e(V_2) \le \Delta/2$.

With the same arguments as that of Case 1, we can show that for $i = 1, 2, \ldots, k - r$,

$$|N(u_i) \cap V_2| + |N(v_i) \cap V_1| \ge |N(u_i) \cap V_1| + |N(v_i) \cap V_2|$$

and for $i = k - r + 1, \ldots, k$,

$$|N(u_i) \cap V_2| + |N(v_i) \cap V_1| \ge |N(u_i) \cap V_1| + |N(v_i) \cap V_2| + 2.$$

By the maximality of $e(V_1, V_2)$, we have

$$|N(y_0) \cap V_p| \le |N(y_0) \cap V_{3-p}|.$$

Thus

$$\begin{aligned} e(V_1, V_2) &= \frac{1}{2}\left(|N(y_0) \cap V_{3-p}| + \sum_{i=1}^{k} (|N(u_i) \cap V_2| + |N(v_i) \cap V_1|) \right) \\ &\ge \frac{1}{2}\left(|N(y_0) \cap V_p| + \sum_{i=1}^{k} (|N(u_i) \cap V_1| + |N(v_i) \cap V_2|) \right) + r \\ &= e(V_1) + e(V_2) + r. \end{aligned}$$

Therefore, since $0 \le e(V_1) - e(V_2) \le \Delta/2$ and $e(V_1) + e(V_2) + e(V_1, V_2) = m$, we have $\max\{e(V_1), e(V_2)\} \le (m - r + \Delta)/4$ and $e(V_1, V_2) \ge (m + r)/2$. ∎

From the proof of Theorem 1.1, one can extract a polynomial time algorithm for finding a balanced bipartition satisfying the bounds in Theorem 1.1.

3 Related results

In this section we give further results on balanced bipartitions. First, we establish Conjecture 1.2 for graphs with minimum degree at least 5.

Corollary 3.1 *Let G be a graph with m edges and $\delta(G) \ge 5$. Then G admits a balanced bipartition V_1, V_2 such that* $\max\{e(V_1), e(V_2)\} \le m/3$.

Proof. Let n denote the number of vertices in G. If $m \ge 3n$ then $\Delta(G) \le n - 1 < m/3$; and it follows from Theroem 1.1 that G admits a balanced bipartition V_1, V_2 such that $\max\{e(V_1), e(V_2)\} \le (m + \Delta)/4 < m/3$. Now assume $m \le 3n - 1$. Then $\delta(G) = 5$.

Let t denote the number of vertices of degree 5, and let $\Delta := \Delta(G)$. We may assume $t \le n - 1$; for otherwise G is a regular graph and the assertion follows from the result of Bollobás and Scott [5].

By Theorem 1.1, G admits a balanced bipartition V_1, V_2 such that $\max\{e(V_1), e(V_2)\} \le (m + \Delta)/4$. If $(m + \Delta)/4 \le m/3$, we are done. So we may assume that $(m + \Delta)/4 > m/3$, i.e., $\Delta > m/3$. Then $2m \ge \Delta + 6(n - t - 1) + 5t > m/3 + 6n - t - 6 \ge m/3 + 5n - 5$ (since $t \le n - 1$). So $m/3 > n - 1$, and hence $\Delta > m/3 > n - 1$, a contradiction. ∎

The next result is also a consequence of Theorem 1.1, which answers a problem of Bollobás and Scott in [3].

Corollary 3.2 *Let G be a graph with m edges and n vertices. Suppose $\Delta(G) = o(n)$, or $\delta(G) \to \infty$ as $n \to \infty$. Then G admits a balanced bipartition V_1, V_2 such that* $\max\{e(V_1), e(V_2)\} \le (1 + o(1))m/4$.

Proof. If $\Delta(G) = o(m)$ then the assertion follows directly from Theorem 1.1. So we may assume that $\Delta(G) = \Omega(m)$ and $\delta(G) \to \infty$ as $n \to \infty$. Then since $\Delta(G) \le n = O(m)$, we must have $m = \Theta(n)$. But this forces $\delta(G) = O(1)$. Therefore, if $\delta(G) \to \infty$ as $n \to \infty$, we must have $\Delta(G) = o(m)$, and hence the assertion follows from Theorem 1.1. ∎

We now give examples that show that the conditions in Corollary 3.2 is best possible in the sense that it does not hold for graphs G with $\Delta(G) = \Theta(n)$ and $\delta(G) = O(1)$. Let $G = K_{n_1,n_2}$, where $n_1 = 2k+1$, $n_1+n_2 = n$, n is even, and $n > 8k$. Let V_1, V_2 be a balanced bipartition of G and, without loss of generality, assume that V_1 contains precisely l vertices of degree $n-(2k+1)$, where $l \ge k+1$. Then $e(V_1) = l(n/2-l)$. Note that the function $f(l) = l(n/2-l)$ is increasing on the interval $[k+1, n/4]$. So $e(V_1) \ge (k+1)(n/2-k-1)$. Note that $m := (2k+1)(n-2k-1)$ is the number of edges in G. Therefore, $e(V_1)/m \ge (k + 1)/(4k + 2) + o(1)$.

These examples also show that $c(k) \ge (k + 1)/(4k + 2) + o(1)$, in the following problem of Bollobás and Scott [3].

Problem 3.3 *What is the smallest constant $c(k)$ such that every graph G with minimal degree at least k has a balanced bipartition with at most $c(k)e(G)$ edges in each class?*

In [3], Bollobás and Scott also posed a question concerning the bound analogous to that of Edwards for maximum bipartite subgraphs: What are the sizes of the largest and smallest bipartite subgraphs that one can guarantee with balanced bipartitions? Here we give a lower bound on the size of a largest balanced bipartite subgraph.

Let G be a regular graph with m edges, and let M be a maximum matching of G. Since G is regular, M is symmetric. So Theorem 1.1 ensures that G admits a balanced bipartition V_1, V_2 such that $e(V_1, V_2) \geq (m+|M|)/2$. Next, we show that this bound holds for non-regular graphs as well.

Theorem 3.4 *Let G be a graph with m edges, and let M be a maximum matching of G. Then G admits a balanced bipartition V_1, V_2 such that $e(V_1, V_2) \geq (m + |M|)/2$.*

Proof. Let $M = \{e_1, e_2, \ldots, e_r\}$ be a maximum matching of G, and let V_1, V_2 be a balanced bipartition of $V(G)$ such that

(1) for each $i \in \{1, 2, \ldots, r\}$, e_i has one end in V_1 and the other in V_2, and

(2) subject to (1), $e(V_1, V_2)$ is maximum.

Let $V_1 = \{u_1, u_2, \ldots, u_s\}$ and $V_2 = \{v_1, v_2, \ldots, v_t\}$, such that $s \geq t$ and $e_i = u_i v_i$ for $i = 1, 2, \ldots, r$.

As in the proof of Theorem 1.1, we can use (1) and (2) to show that for $i = 1, \ldots, r$,

$$|N(u_i) \cap V_2| + |N(v_i) \cap V_1| \geq |N(u_i) \cap V_1| + |N(v_i) \cap V_2| + 2$$

and for $i = r + 1, \ldots, t$,

$$|N(u_i) \cap V_2| + |N(v_i) \cap V_1| \geq |N(u_i) \cap V_1| + |N(v_i) \cap V_2|.$$

Moreover, if $s = t + 1$ then $|N(u_s) \cap V_2| \geq |N(u_s) \cap V_1|$.

Suppose $s = t$. Then

$$\begin{aligned} 2m &= \sum_{i=1}^{t}(|N(u_i) \cap V_2| + |N(v_i) \cap V_1|) + \sum_{i=1}^{t}(|N(u_i) \cap V_1| + |N(v_i) \cap V_2|) \\ &\leq 2\sum_{i=1}^{t}|N(u_i) \cap V_2| + 2\sum_{i=1}^{t}|N(v_i) \cap V_1|) - 2r \\ &= 4e(V_1, V_2) - 2r. \end{aligned}$$

Hence, $e(V_1, V_2) \geq (m + |M|)/2$.

Now assume $s = t + 1$. Then

$$\begin{aligned} 2m &= \sum_{i=1}^{s}|N(u_i) \cap V_2| + \sum_{i=1}^{s}|N(u_i) \cap V_1| + \sum_{i=1}^{t}|N(v_i) \cap V_1| + \sum_{i=1}^{t}|N(v_i) \cap V_2| \\ &\leq 2\sum_{i=1}^{s}|N(u_i) \cap V_2| + 2\sum_{i=1}^{t}|N(v_i) \cap V_1| - 2r \\ &= 4e(V_1, V_2) - 2r. \end{aligned}$$

Again, $e(V_1, V_2) \geq (m + |M|)/2$ as required. ∎

It is easy to see that the bound in Theorem 3.4 is best possible for complete graphs. Since a maximum matching can be found in polynomial time [6], our proof implies a polynomial time algorithm for finding a balanced bipartition as described in Theorem 3.4.

Theorem 3.4 gives a lower bound on the maximum size of a balanced bipartite subgraph in terms of the size of a maximum matching. A natural question is the following formulation of the Bollobás-Scott question mentioned above.

Problem 3.5 *Is there a function $f(m)$ such that every graph with m edges admits a balanced bipartition having at least $m/2 + f(m)$ edges between the two subsets?*

In [14], it is proved that every graph G with $\Delta(G) \leq \frac{7}{5}\delta(G)$ admits a balance bipartition V_1, V_2 such that $\max\{e(V_1), e(V_2)\} \leq |E(G)|/3$. Its proof actually shows that for any graph G with $\Delta(G) \leq \frac{7}{5}\delta(G)$, any balanced bipartition V_1, V_2 of $V(G)$ with $e(V_1, V_2)$ maximum (among all balanced bipartitions) must satisfy $e(V_i) \leq |E(G)|/3$. Theorem 3.4 ensures that the balanced bipartition V_1, V_2 constructed in [14] has $e(V_1, V_2) \geq (|E(G)| + m(G))/2$, where $m(G)$ denotes the number of edges in a maximum matching of G. An example given in [14] shows that there exist graphs G such that for any maximum balanced bipartition V_1, V_2 of $V(G)$, $\max\{e(V_1), e(V_2)\} > |E(G)|/3$. So it would be interesting to know which graphs G admits a maximum balanced bipartition V_1, V_2 such that
$\max\{e(V_1), e(V_2)\} \leq |E(G)|/3$?

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