On the Reconstruction of Planar Graphs

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Abstract

We show that the planarity of a graph can be recognized from its vertex deleted subgraphs, which answers a question posed by Bondy and Hemminger in 1979. We also state some useful counting lemmas and use them to reconstruct certain planar graphs.

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1 Introduction

All graphs considered in this paper are simple, and unless otherwise specified we shall use the terminology of Diestel [3]. For a graph G, a graph H is said to be a *reconstruction* of G if there is a bijection $\sigma : V(G) \to V(H)$ such that $G - v \cong H - \sigma(v)$ for all $v \in V(G)$. The graphs $G - v$ are the *vertex deleted subgraphs* of G. A graph G is said to be *reconstructible* if every reconstruction of G is isomorphic to G. A parameter $t(G)$ is said to be *reconstructible* if $t(H) = t(G)$ for all reconstructions H of G.

In 1942, Ulam and Kelly ([18], [11], also see [1]) made the Vertex Reconstruction Conjecture: Every simple graph with at least three vertices is reconstructible. There is also an edge version known as the Edge Reconstruction Conjecture made by Harary [9]: If G and H are simple graphs with at least four edges and there is a bijection $\sigma : E(G) \to E(H)$ such that $G - e \cong H - \sigma(e)$ for all $e \in E(G)$, then $G \cong H$. For an account of known results, we refer the readers to [2], [4], and [1].

There are two types of results concerning vertex reconstruction. One type involves the reconstruction of graph parameters. For example, Tutte [17] proved that the chromatic number of a graph is reconstructible and the number of 2-connected spanning subgraphs with a fixed number of edges (including Hamilton cycles) is reconstructible. Another type involves reconstructing classes of graphs. For example, Kelly [12] proved that disconnected graphs, regular graphs, and trees are reconstructible. Before reconstructing a class of graphs, one usually needs to recognize this class. A class $\mathcal{G}$ of graphs is said to be *recognizable* if for each $G \in \mathcal{G}$, every reconstruction of G is also in $\mathcal{G}$.

Fiorini [5] proved that the class of planar graphs with minimum degree at least 5 is recognizable by showing that a graph with minimum degree at least 5 is planar iff $G - v$ is planar for all $v \in V(G)$. In fact, this follows from a result of Wagner [19] which characterizes all nearly planar graphs, i.e., those non-planar graphs G for which $G - v$ is planar for all $v \in V(G)$. However, it is not clear how to apply Wagner's result to recognize the class of all planar graphs. Fiorini and Manvel [7] characterized all nearly planar graphs of minimum degree 4, and proved that the class of planar graphs with minimum degree at least 4 is recognizable. Further, Fiorini and Lauri [6] showed that the class of maximal planar graphs is recognizable, without giving a complete characterization of graphs G for which $G - v$ is "nearly" maximal planar for all $v \in V(G)$. Lauri [15] then showed that all maximal planar graphs are reconstructible.

In his notes [10], Hemminger posed the problem of recognizing the class of planar graphs with minimum degree at least 3. Bondy and Hemminger in [2] (p. 236) mentioned the problem of recognizing the class of planar graphs, and they noted that recognizing the class of planar graphs is just a special case of the problem to reconstruct the genus of a graph. The main result of this paper is the following.

(1.1) Theorem. *The class of planar graphs is recognizable.*

To prove Theorem (1.1), we need to prove several counting lemmas. We believe that these counting lemmas should be useful for reconstruction, and as evidence we use them to reconstruct certain 5-connected planar graphs. It is still an open problem [2] to reconstruct all planar graphs.

2 Counting lemmas

Let G, X be graphs, and let $s(X, G)$ denote the number of subgraphs of G isomorphic to X. Kelly [12] showed that when $|V(X)| < |V(G)|$ then $s(X, G)$ is reconstructible. To recognize planar graphs, we need to reconstruct the number of certain induced subgraphs. Let $X_1, \ldots, X_n, G$ be graphs, and let k be a positive integer. We use $s_k((X_1, \ldots, X_n), G)$ to denote the number of sequences $(G_1, \ldots, G_n)$ of induced subgraphs of G such that $|V(\bigcup_{i=1}^n G_i)| = k$ and $G_i \cong X_i$ for all $1 \leq i \leq n$. When $n = 1$, $s_k((X_1, \ldots, X_n), G)$ is simply denoted $s_k(X_1, G)$, which is the number of k-vertex induced subgraphs of G isomorphic to X_1. The following result for $n = 2$ is given in [14]; its proof below is implicit in [14] and simlar to that of its non-induced version in [1].

(2.1) Lemma. *Let $X_1, \ldots, X_n, G$ be graphs, and let k be a positive integer. Suppose $k < |V(G)|$. Then $s_k((X_1, \ldots, X_n), G)$ is reconstructible.*

Proof. Let $(G_1, \ldots, G_n)$ be a sequence of induced subgraphs of G such that $|V(\bigcup_{i=1}^n G_i)| = k$. Because $k < |V(G)|$, $(G_1, \ldots, G_n)$ occurs in exactly $|V(G)| - k$ vertex deleted subgraphs of G. Hence,

$$s_k((X_1, \ldots, X_n), G) = \frac{\sum_{v \in V(G)} s_k((X_1, \ldots, X_n), G - v)}{|V(G)| - k}.$$

It is then easy to see that $s_k((X_1, \ldots, X_n), G)$ is reconstructible. □

Kocay [13] proved a simple and yet powerful lemma which was used to reconstruct the number of certain spanning subgraphs. Let $\mathcal{F} = (F_1, ..., F_n)$ be a sequence of graphs, not necessarily distinct. A cover of a graph G by $\mathcal{F}$ is a sequence $(G_1, \ldots, G_n)$ of subgraphs of G such that $\bigcup_{i=1}^n G_i = G$ and $G_i \cong F_i$ for all $1 \leq i \leq n$. Let $c(\mathcal{F}, G)$ denote the number of covers of G by $\mathcal{F}$. Kocay showed that if $|V(F_i)| < |V(G)|$ for all $1 \leq i \leq n$ then $c(\mathcal{F}, G)$ is reconstructible. In [14] Kocay also proved an induced version. Let $\mathcal{F} = (F_1, ..., F_n)$ be a sequence of graphs, not necessarily distinct. An *induced vertex cover* of a graph G by $\mathcal{F}$ is a sequence $(G_1, \ldots, G_n)$ of induced subgraphs of G such that $\bigcup_{i=1}^n V(G_i) = V(G)$ and $G_i \cong F_i$ for all $1 \leq i \leq n$. Let $\widetilde{c}(\mathcal{F}, G)$ denote the number of induced covers of G by $\mathcal{F}$. The following result for $n = 2$ is given in [14]; its proof below is implicit in [14] and simlar to that of its non-induced version in [1].

(2.2) Lemma. *Let $\mathcal{F} = (F_1, ..., F_n)$ be a sequence of graphs, not necessarily distinct, let G be a graph, and assume that $|V(F_i)| < |V(G)|$ for all $1 \leq i \leq n$. Then $\widetilde{c}(\mathcal{F}, G)$ is reconstructible.*

Proof. By counting the number of sequences $(G_1, \ldots, G_n)$ of induced subgraphs of G such that $G_i \cong F_i$ for all $1 \leq i \leq n$, we have

$$\prod_{j=1}^{n} s_{|V(F_j)|}(F_j, G) = \sum_{k=1}^{|V(G)|} s_k((F_1, \ldots, F_n), G).$$

Because $|V(F_j)| < |V(G)|$, it follows from Lemma (2.1) that $s_{|V(F_j)|}(F_j, G)$ is reconstructible. Hence $\prod_{j=1}^n s_{|V(F_j)|}(F_j, G)$ is reconstructible. Therefore, $\sum_{k=1}^{|V(G)|} s_k((F_1, \ldots, F_n), G)$ is also reconstructible. By Lemma (2.1) again, $\sum_{k=1}^{|V(G)|-1} s_k((F_1,\ldots,F_n),G)$ is reconstructible. Hence, $\widetilde{c}(\mathcal{F},G) = s_{|V(G)|}((F_1,\ldots,F_n),G)$ is also reconstructible. □

To recognize planar graphs, we shall use Lemma (2.2) to count the number of induced non-separating cycles in a graph G. (A subgraph H of a connected graph G is said to be *non-separating* if $G-V(H)$ is connected.) This will be done by considering induced vertex covers (F_1,F_2) of G, where F_1 is a cycle, and F_2 is a connected graph on $|V(G)|-|V(F_1)|$ vertices. The conditions on (F_1,F_2) will ensure that the cycle in G corresponding to F_1 is non-separating in G.

For the purpose of reconstructing certain planar graphs, we also need to show that the number of certain local structures in those planar graphs are reconstructible. By a *vertex wheel* we mean a graph obtained from a cycle C by adding a vertex v and at least three edges from v to C. (A vertex wheel is necessarily planar.) The vertex v is the *center* of the wheel, and the cycle C is the *rim* of the wheel. Note that a vertex wheel has at most one vertex of degree 4 or more, namely, the center.

Although a theorem of Whitney (see [3]) tells us that 3-connected planar graphs have unique planar embeddings, we prefer to work with plane graphs because we shall refer to the faces and facial cycles in actual planar embeddings.

Let G be a 3-connected plane graph. Let $v \in V(G)$ and define the *wheel neighborhood* of v in G, denoted $W_G(v)$, as the union of all facial cycles of G containing v. Let $v_1,\ldots,v_k$ denote the neighbors of v in cyclic order around v, and let F_i, $i=1,\ldots,k$, denote the facial cycles of G containing $\{v,v_i,v_{i+1}\}$, where $v_{k+1}=v_1$. Then since G is 3-connected, $\bigcup_{i=1}^{k}(F_i - v_i)$ is a cycle in G. Hence, $W_G(v)$ is a vertex wheel.

The plane graph G in Figure 1 is 3-connected. Clearly $G-y$ is a vertex wheel with center v and is non-separating. However, the wheel neighborhood of v is $W_G(v) = G-ux$, and clearly $W_G(v) \neq G-y$. In the proof below, we shall see that in 4-connected plane graphs, non-separating induced vertex wheels are wheel neighborhoods. This in turn allows us to use Lemma (2.2) to count the number of wheel neighborhoods.

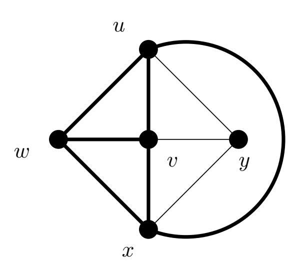

Figure 1: A vertex wheel that is not a wheel neighborhood.

(2.3) Lemma. *Let G be a 4-connected plane graph and let W be a vertex wheel. Then $|\{W_G(v) : v \in V(G)$ and $W_G(v) \cong W\}|$ is reconstructible.*

Proof. First, we observe that in a 4-connected plane graph, any wheel neighborhood of a vertex is non-separating. For otherwise, let $v \in V(G)$, let C denote the rim of $W_G(v)$, and let $C_1,\ldots,C_k$ ($k \geq 2$) denote the components of $G - V(W_G(v))$. Then by planarity, there exist two vertices u,w on C such that for some $1 \leq i \leq k$, all neighbors of C_i on C are contained in one subpath of C between u and w, and all neighbors of other C_j,

$j \neq i$, on C are contained in the other subpath of C between u and w. Then $\{u, v, w\}$ is a 3-cut in G, contradicting the assumption that G is 4-connected.

Second, we observe that if an induced non-separating subgraph K of G is isomorphic to a vertex wheel, then K must be the wheel neighborhood of some vertex in G. To see this, let C and v denote the rim and center, respectively, of some wheel representation of K. Since K is non-separating, there is only one component of $G - V(K)$, denoted D. By planarity of G, D must be contained entirely in a face of K. In fact, D must be contained in the face of K bounded by C; for otherwise, some vertex on C would have degree at most three in G, contradicting the assumption that G is 4-connected. It is then easy to see that K must be the wheel neighborhood of v in G.

With the above observations, we see that by mapping $W_G(v)$ to $(W_G(v), G - V(W_G(v)))$, we have a bijection between $\{W_G(v) : v \in V(G) \text{ and } W_G(v) \cong W\}$ and $\{(F_1, F_2) : (F_1, F_2) \text{ is an induced vertex cover of } G,\ F_1 \cong W,\ |V(F_1)| + |V(F_2)| = |V(G)|, \text{ and } F_2 \text{ is connected}\}$. Further, $|W_G(v)| < |V(G)|$ as G is 4-connected and $W_G(v)$ is an induced subgraph of G with minimum degree 3. Thus $|\{W_G(v) : v \in V(G) \text{ and } W_G(v) \cong W\}| = \widetilde{c}((W_G(v), G - V(W_G(v))), G)$, which is reconstructible by Lemma (2.2). □

By an *edge wheel* we mean a planar graph obtained from a cycle C by adding two adjacent vertices u and v and at least two edges from each of $\{u, v\}$ to C. Again, C is the *rim* of the wheel and uv is the *center* of the wheel. Note that the vertices incident with the *center* are the only vertices whose degrees may be 5 or higher.

Let G be a 4-connected plane graph, and let $e = uv$ be an edge of G. The *wheel neighborhood* of e in G, denoted $W_G(e)$, is the union of all facial cycles of G incident with u or v or both. By a similar argument as for vertex wheels, we can show that $W_G(e)$ is an edge wheel (because G is 4-connected).

(2.4) Lemma. *Let G be a 5-connected plane graph and let W be an edge wheel. Then $|\{W_G(e) : e \in E(G) \text{ and } W_G(e) \cong W\}|$ is reconstructible.*

Proof. The proof is very similar to that of Lemma (2.3), and hence, we give only an outline. First, we observe that in a 5-connected plane graph, any wheel neighborhood of an edge is non-separating. Second, we observe that if an induced non-separating subgraph of a 5-connected plane graph is isomorphic to an edge wheel, then it must be the wheel neighborhood of some edge. Therefore, we can establish a bijection between $\{W_G(e) : e \in E(G) \text{ and } W_G(e) \cong W\}$ and $\{(F_1, F_2) : (F_1, F_2) \text{ is an induced vertex cover of } G,\ F_1 \cong W,\ |V(F_1)| + |V(F_2)| = |V(G)|, \text{ and } F_2 \text{ is connected}\}$, and hence it follows from Lemma (2.2) that $|\{W_G(e) : e \in E(G) \text{ and } W_G(e) \cong W\}|$ is reconstructible. □

Note that the assumption in Lemma (2.4) that G is 5-connected is necessary as the graph in Figure 2(a) shows a 4-connected plane graph G with $W_G(uv)$ separating.

Similarly, we can define a *cycle wheel* as a planar graph obtained from a cycle D by adding a cycle C and at least one edge from each vertex of D to C. Again, C is the *rim* of the wheel, and D is the *center* of the wheel.

Let G be a 4-connected plane graph, and let F be a facial cycle of G. The *wheel neighborhood* of F in G, denoted $W_G(F)$, is the union of all facial cycles of G incident with a vertex of F (including F itself). By a similar argument for edge wheels, we can show $W_G(F)$ is a cycle wheel.

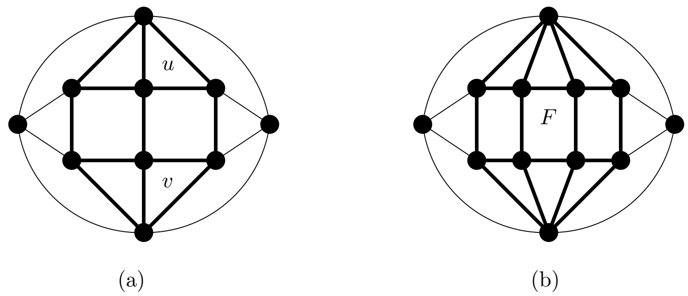

Figure 2: 4-Connected examples

(2.5) Lemma. *Let G be a 5-connected plane graph and let W be a cycle wheel. Then $|\{W_G(F) : F$ is a facial cycle and $W_G(F) \cong W\}|$ is reconstructible.*

Proof. Again the proof is similar to that of Lemma (2.3). First, we observe that in a 5-connected plane graph, any wheel neighborhood of a facial cycle is non-separating. Second, we observe that if an induced non-separating subgraph of a 5-connected plane graph isomorphic to a cycle wheel, then it must be the wheel neighborhood of some facial cycle. Therefore, we can establish a bijection between $\{W_G(F) : F$ is a facial cycle of G and $W_G(F) \cong W\}$ and $\{(F_1, F_2) : (F_1, F_2)$ is an induced vertex cover of G, $F_1 \cong W$, $|V(F_1)|+|V(F_2)| = |V(G)|$, and F_2 is connected$\}$, and hence it follows from Lemma (2.2) that $|\{W_G(F) : F$ is a facial cycle and $W_G(F) \cong W\}|$ is reconstructible. □

Again, 5-connectedness in Lemma (2.5) is necessary as the graph in Figure 2(b) shows a 4-connected plane graph G with $W_G(F)$ separating.

3 Recognizing planar graphs

We give a proof of Theorem (1.1) in this section. Since by definition a graph G is k-connected iff $|V(G)| \geq k+1$ and G admits no vertex cut of size at most $k-1$, G is k-connected iff $|V(G)| \geq k+1$ and, for each $v \in V(G)$, $G-v$ admits no vertex cut of size at most $k-2$. So for $k \geq 2$, k-connectivity is recognizable. However, since disconnected graphs are reconstructible, the connectivity of a graph is reconstructible.

(3.1) Lemma. *Let G be a planar graph which is connected but not 2-connected, and let H be a reconstruction of G. Then H is also planar.*

Proof. Because the connectivity of G is reconstructible, H is connected but not 2-connected. Let v be a cutvertex of H and let $C_1, ..., C_k$ be the components of $G-v$, where $k \geq 2$. Define H_1 to be the subgraph of H induced by $V(C_1) \cup \{v\}$, and let H_2 be the subgraph of H induced by $(\bigcup_{i=2}^{k} V(C_i)) \cup \{v\}$. Then $V(H_1 \cap H_2) = \{v\}$, $|V(H_i)| < |V(H)|$ for $i \in \{1,2\}$, and $H_1 \cup H_2 = H$.

Note that $s_{|V(H_i)|}(H_i, H) > 0$ for $i \in \{1,2\}$. By Lemma (2.1), $s_{|V(H_i)|}(H_i, H)$ is reconstructible. Hence, $s_{|V(H_i)|}(H_i, G) > 0$. Therefore, since G is planar, H_i is planar. It is then easy to see that H is planar. □

For the recognition of all planar graphs, we need to reconstruct the number of induced non-separating cycles in connected graphs. For any $3 \leq k \leq |V(G)|$, let $n(G,k)$ denote the number of induced non-separating cycles of length k in G.

(3.2) Lemma. *Let G be a connected graph and let k be an integer with $3 \leq k \leq |V(G)|$. Then $n(G,k)$ is reconstructible.*

Proof. We observe that if C is an induced non-separating cycle in G, then $(C, G-V(C))$ is an induced vertex cover of G in which $|V(C)| + |V(G-V(C))| = |V(G)|$, C is a cycle, and $G - V(C)$ is connected. Conversely, if (F_1, F_2) is an induced vertex cover of G in which $|V(F_1)| + |V(F_2)| = |V(G)|$, F_1 is a cycle, and F_2 is connected, then F_1 corresponds to an induced non-separating cycle in G. Hence, it is easy to see that there is a natural bijection, as indicated above, between $\{C : C$ is an induced non-separating cycle of G and $|V(C)| = k\}$ and $\{(F_1, F_2) : (F_1, F_2)$ is an induced vertex cover of G, F_1 is a cycle, $|V(F_1)| = k$, $|V(F_1)| + |V(F_2)| = |V(G)|$, and F_2 is connected$\}$. So $n(G,k) = \sum_{F_2} \widetilde{c}((F_1,F_2),G)$, where F_1 is a cycle of length k, F_2 is connected, $|V(F_1)| + |V(F_2)| = |V(G)|$, and the summation is over all isomorphism types of connected graphs F_2 on $|V(G)| - k$ vertices. Thus by Lemma (2.2), $n(G,k)$ is reconstructible for $3 \leq k < |V(G)|$. Further, $n(G, |V(G)|) = 0$ unless G is a cycle (in which case G is reconstructible). □

A *subdivision* of a graph K is a graph obtained from K by replacing edges of K with pairwise internally disjoint paths. If K is a subdivision of a 3-connected graph, then either K is 3-connected, or for any 2-cut $\{u,v\}$ of K, $uv \notin E(K)$, $K - \{u,v\}$ has precisely two components exactly one of which, say D, is such that the subgraph of K induced by $V(D) \cup \{u,v\}$ is a path. This observation is used below to recognize subdivisions of 3-connected graphs.

(3.3) Lemma. *The class of subdivisions of 3-connected graphs is recognizable.*

Proof. Let G be a subdivision of a 3-connected graph. Clearly, G is 2-connected. Let H be a reconstruction of G, and we wish to show that H is a subdivision of a 3-connected graph. Since connectivity is reconstructible, H is 2-connected. If H is 3-connected then H is a subdivision of a 3-connected graph. Therefore, we may assume that H is not 3-connected. Further, since G is a subdivision of a 3-connected graph and degree sequence is reconstructible, H has at least four vertices of degree at least 3.

Suppose for a contradiction that H is not a subdivision of a 3-connected graph. Then H has a 2-cut $\{u,v\}$ such that either $uv \in E(H)$, or $H - \{u,v\}$ has at least three components, or $H - \{u,v\}$ has exactly two components C_1 and C_2, and for each i the subgraph of H induced by $V(C_i) \cup \{u,v\}$ is not a path. Let $C_1, ..., C_k$, $k \geq 2$, be the components of $H - \{u,v\}$. Since H has at least four vertices of degree at least 3, we may assume without loss of generality that H_1, the subgraph of H induced by $V(C_1) \cup \{u,v\}$, has a vertex of degree ≥ 3. Let H_2 be the subgraph of H induced by $(\bigcup_{i=2}^{k} V(C_i)) \cup \{u,v\}$. Then $|V(H_j)| < |V(G)|$ $(j = 1,2)$, $H_1 \cup H_2 = H$, $V(H_1 \cap H_2) = \{u,v\}$, and neither H_1 nor H_2 is path.

Note, (H_1, H_2) forms an induced cover of H. By Lemma (2.2), G has an induced cover (G_1, G_2) where $G_1 \cong H_1$ and $G_2 \cong H_2$. Suppose $uv \in E(H)$. Since $|V(G)| = |V(H)|$ and $|E(G)| = |E(H)|$, $G_1 \cap G_2$ consists of two vertices and a single edge. This contradicts the assumption that G is a subdivision of a 3-connected graph. Hence $uv \notin E(H)$. By counting vertices and edges again, $G_1 \cap G_2$ consists of two vertices and no edges. As these two vertices form a 2-cut in G and neither G_1 nor G_2 is a path, we again have a contradiction. □

A classical result of Tutte [16] states that a 3-connected graph G is planar iff every edge of G is contained in exactly two induced non-separating cycles in G. We observe that the same result holds for subdivisions of 3-connected graphs. A *branch* path in a subdivision G of a 3-connected graph is a path whose ends have degree at least three in G and whose internal vertices have degree two in G.

(3.4) Lemma. *A subdivision G of a 3-connected graph is planar iff every edge of G is contained in exactly two induced non-separating cycles in G.*

Proof. To see this, let H denote the 3-connected graph such that G is a subdivision of H. Given a cycle D in G, let D' denote the cycle in H obtained from D by replacing branch paths of G contained in D with edges. Clearly, D is induced and non-separating in G iff D' is induced and non-separating in H. Therefore, the assertion of this lemma follows from the above mentioned theorem of Tutte. □

Tutte [16] also showed that for any 3-connected graph G and any edge e of G, there are at least two induced non-separating cycles in G containing e. The same proof for Lemma (3.4) proves the following as well.

(3.5) Lemma. *For any subdivision G of a 3-connected graph and any edge e of G, there are at least two induced non-separating cycles in G containing e.*

With help from the above lemmas, we can recognize planarity of subdivisions of 3-connected graphs.

(3.6) Lemma. *Let G be a subdivision of a 3-connected planar graph, and let H be a reconstruction of G. Then H is planar.*

Proof. Since G is a subdivision of a 3-connected graph, it follows from Lemma (3.3) that H is also a subdivision of a 3-connected graph. We wish to show that H is planar.

By Lemma (3.2), $n(G,k) = n(H,k)$ for all $3 \leq k \leq |V(G)|$. By Lemma (3.4), $\sum_{k=3}^{n} n(G,k)k = 2|E(G)|$. Since H is a subdivision of a 3-connected graph, it follows from Lemma (3.5) that $\sum_{k=3}^{n} n(H,k)k \geq 2|E(H)|$. Since $|E(G)| = |E(H)|$ and $n(G,k) = n(H,k)$ for all $3 \leq k \leq |V(G)|$, we have $\sum_{k=3}^{n} n(H,k)k = 2|E(H)|$. This implies that every edge of H is contained in exactly two induced non-separating cycles in H. Now it follows from Lemma (3.4) that H is planar. □

For planar graphs which are not 3-connected, we shall see that the recognition problem can be reduced to that for subdivisions of 3-connected graphs.

(3.7) Theorem. *The class of planar graphs is recognizable.*

Proof. Let G be a planar graph and let H be a reconstruction of G. If H is not connected or H is a cycle, then H is reconstructible, and so, H is planar. If H is connected but not 2-connected, then it follows from Lemma (3.1) that H is planar. If H is a subdivision of a 3-connected graph, then H is planar by Lemma (3.6).

Therefore we may assume that H is 2-connected and H is neither a cycle nor a subdivision of a 3-connected graph. Thus, H has a 2-cut $\{u,v\}$ such that $uv \in E(H)$, or $H - \{u,v\}$ has at least 3 components, or $H - \{u,v\}$ has exactly two components C_1 and

C_2, and the subgraph of H induced by $V(C_i) \cup \{u, v\}$ is a path for $i = 1, 2$. Let C_1,...,C_l, $l \geq 2$, denote the components of $H - \{u, v\}$. Let H_i be the subgraph of H obtained from C_i by adding $\{u, v\}$ and all the edges of G from $\{u, v\}$ to $V(C_i)$. Let $k = l$ if $uv \notin E(H)$, and otherwise let $k = l + 1$ and H_k be the subgraph of H induced by $\{u, v\}$. Thus, either $k \geq 3$, or $k = 2$ and neither H_1 nor H_2 is a path. Further, $|V(H_i)| < |V(H)|$ for $1 \leq i \leq k$.

Clearly, for each $1 \leq i \leq k$, $s(H_i, H) > 0$. Since $|V(H_i)| < |V(G)|$, it follows from Kelly's lemma that $s(H_i, H)$ is reconstructible. Hence $s(H_i, G) > 0$, which implies that H_i is planar.

If $H_i + uv$ is planar for all $1 \leq i \leq k$ then we see that each H_i has a planar embedding such that u and v are incident with its infinite face. For any $i < k$, we can draw $H_{i+1} + uv$ inside a finite face of $H_i + uv$ incident with uv. Consequently we can obtain a planar embedding of $H + uv$, and hence H is planar.

Thus, we may assume that some $H_i + uv$ is not planar. By Kuratowski's theorem, $H_i + uv$ contains a subgraph K which is isomorphic to a subdivision of K_5 or $K_{3,3}$.

Suppose $uv \notin E(K)$. Then K is a proper subgraph of H, and so, $s(K, H) > 0$. Therefore, by Kelly's lemma, $s(K, G) > 0$. This is a contradiction, because G is planar. So $uv \in K$. Since either $k \geq 3$, or $k = 2$ and neither H_1 nor H_2 is a path, we see that there is a path P in $H - V(H_i - \{u, v\})$ between u and v such that $V(P) \neq V(H) - V(H_i - \{u, v\})$. Let K' be the subgraph of H obtained from K by replacing uv with P. Then K' is also a subdivision of K_5 or $K_{3,3}$. Clearly, $|V(K')| < |V(H)|$ and $s(K', H) > 0$. Hence by Kelly's lemma, $s(K', G) > 0$. Again, this contradicts the planarity of G. □

4 Reconstruction

In general, vertex reconstruction is difficult. Here, we apply results obtained in previous sections to reconstruct certain 5-connected planar graphs. We shall focus on vertices of minimum degree. By Euler's formula, the minimum degree of a 5-connected plane graph is 5, and our strategy is to consider how far apart these vertices are. One extreme case is when all neighbors of a vertex have degree 5, and in that case the graph is easily seen reconstructible because the degree sequence of a graph is reconstructible. The other extreme situation is when there is a degree 5 vertex that is far away from all other degree 5 vertices. However it turns out that this case is not trivial, and we deal with it in the remainder of this paper.

Let G be a 5-connected plane graph. For any $u \in V(G)$, the *new face* of $G - u$ is the face of $G - u$ that is not a face of G. A vertex of degree 5 is simply called a *5-vertex*, and an edge incident with two 5-vertices is called a *5-edge*. We say that a 5-vertex v is *isolated* if for any other 5-vertex u, no vertex of $W_G(v)$ is cofacial with any vertex of $W_G(u)$. *Isolated* 5-edges are defined in the same way.

(4.1) Proposition. *Let G be a 5-connected plane graph, and assume that G has an isolated 5-edge. Then G is reconstructible.*

Proof. Let uw be an isolated 5-edge in G, let w, u_1, u_2, u_3, u_4 be the neighbors of u in counter-clockwise order around u, and let u, w_1, w_2, w_3, w_4 be the neighbors of w in counter-clockwise order around w. Let F and T be the facial cycles of G containing

$\{u, w, u_1, w_4\}$ and $\{u, w, u_4, w_1\}$, respectively. Let $F, F_1, F_2, F_3, T, T_1, T_2, T_3$ be the facial cycles of G containing u or w or both, which occur in counter-clockwise order around uw. See Figure 3.

Without loss of generality, we may assume that $|V(F)| \geq |V(T)|$. Moreover, if possible, when $|V(F)| = |V(T)| = 3$ we may assume the notation is chosen so that $|V(F_1)| = |V(T_3)|$.

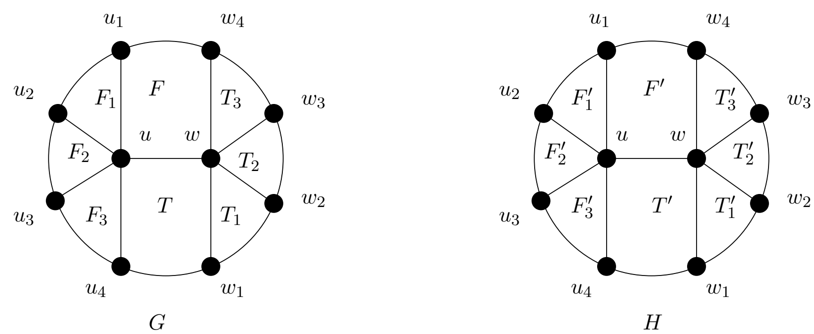

Figure 3: $W_G(uv)$ and $W_H(uv)$

Let H be a reconstruction of G obtained from $G - u_1$ by adding u_1 back to $G - u_1$. Because u has the minimum degree 5 in G and degree sequence is reconstructible, u_1 must be added to a face of $G - u_1$ containing u. Since uw is an isolated 5-edge, $W_G(uw)$ is the only wheel neighborhood of a 5-edge in G which may be changed in H. So $W_G(uw) \cong W_H(uw)$ by Lemma (2.4). By the assumption that $|V(F)| \geq |V(T)|$, we must add u_1 into the new face of $G - u_1$.

We use the same notation as in G for all vertices of H, and we use $F', F_1', F_2', F_3', T', T_1', T_2', T_3'$ to denote the facial cycles of H containing u or w which occur around uw in counter-clockwise order, starting from the one containing $\{u, w, u_1, w_4\}$.

Suppose that $|V(F)| \geq 4$. Since $W_G(uw) \cong W_H(uw)$, $|V(F')| = |V(F)|$ and $|V(F_1')| = |V(F_1)|$, which implies that $W_G(T)$ remains unchanged in H. Since $W_G(F)$ is the only wheel neighborhood of a facial cycle of G containing a 5-edge in G that may be changed in H, $W_G(F) \cong W_H(F')$ by Lemma (2.5). Let $|V(F)|, a_1, a_2, \ldots, a_k$ be the sizes of facial cycles of G containing u_1 which occur in counter-clockwise order around u_1 (starting from F), let $|V(F)|, b_1, b_2, \ldots, b_t$ be the sizes of facial cycles of G containing w_4 which occur in clockwise order around w_4 (starting from F), and let $|V(F')|, c_1, c_2, \ldots, c_k$ be the sizes of facial cycles of H containing u_1 which occur in counter-clockwise order around u_1 (starting from F'). Note that any isomorphism $\pi : W_G(F) \to W_H(F')$ must send u_1, w_4 to u_1, w_4, respectively, or to w_4, u_1, respectively, because π sends F to F' and uw is an isolated 5-edge. If the former case holds then $H \cong G$. So we may assume $\pi(u_1) = w_4$ and $\pi(w_4) = u_1$. Then, $k = t$ and for each $1 \leq i \leq k$, $a_i = b_i$ and $b_i = c_i$. It implies that $a_i = c_i$, and again, $H \cong G$.

Therefore, we may assume $|V(F)| = 3$. Then $|V(T)| = 3$ by the choice of F.

Suppose $|V(F_1)| = |V(T_3)|$. Then, since $W_G(uw) \cong W_H(uw)$, $|V(F')| = |V(F)|$ and $|V(F_1')| = |V(F_1)| = |V(T_3)| = |V(T_3')|$. Therefore $W_G(T)$ remains unchanged in H. By Lemma (2.5), $W_G(F) \cong W_H(F')$. Note that each isomorphism $\pi : W_G(F) \to W_H(F')$ must send u, w to u, w, respectively, or to w, u respectively (since π sends F to F' and uw is an isolated 5-edge in G). If the former case holds then $H \cong G$. We may therefore assume that $\pi(u) = w$ and $\pi(w) = u$. Thus $\pi(F_2) = T_2'$ and $\pi(T_2) = F_2'$, and hence $|V(F_2')| = |V(F_2)| = |V(T_2)| = |V(T_2')|$. In addition, π sends F_1, F_3, T_1, T_3 to either T_3', T_1', F_3', F_1' or T_1', T_3', F_1', F_3'. If the former case holds, then $|V(F_1')| = |V(T_3)| = |V(T_3')|$ and $|V(F_3')| = |V(T_1)| = |V(T_1')|$. If the latter case holds, then $|V(F_1')| = |V(T_1)| = |V(T_1')| = |V(F_1)| = |V(T_3)| = |V(T_3')|$ and $|V(F_3')| = |V(T_3)| = |V(F_1)| = |V(T_1')|$. Hence in either case, $|V(F_1')| = |V(T_3')|$ and $|V(F_3')| = |V(T_1')|$. Next, we shall show that H is reconstructible. Let K be a reconstruction obtained from $H - u$ by adding u back. Since degree sequence is reconstructible and w is a 5-vertex in G, u should be inserted to some face of $H - u$ containing w. Since the size of the new facial cycle of $H - u$ is greater than the sizes of other facial cycles of $H - u$ containing w, u should be added to the new face of $H - u$. Again, we use the same notation as in H for vertices of K. Let $F'', F_1'', F_2'', F_3'', T'', T_1'', T_2'', T_3''$ denote the facial cycles of K containing u or w which occur around uw in counter-clockwise order, starting from the one containing $\{u, w, u_1, w_4\}$. Since uw is an isolated 5-edge in H and because degree sequence is reconstructible, we have $W_H(uw) \cong W_K(uw)$. So $|V(F'')| = |V(T'')| = 3$ and $|V(F_2'')| = |V(F_2')| = |V(T_2')|$. Therefore, $|V(F_1'')| = |V(F_1')|$ and $|V(F_3'')| = |V(F_3')|$ or $|V(F_1'')| = |V(F_3')|$ and $|V(F_3'')| = |V(F_1')|$. If $|V(F_1'')| = |V(F_1')|$ and $|V(F_3'')| = |V(F_3')|$ then $K \cong H$. So, we may assume that $|V(F_1'')| \neq |V(F_1')|$ or $|V(F_3'')| \neq |V(F_3')|$, and $|V(F_1'')| = |V(F_3')|$ and $|V(F_3'')| = |V(F_1')|$. Then since $|V(F_1')| = |V(T_3')| = |V(T_3'')|$ and $|V(F_3')| = |V(T_1')| = |V(T_1'')|$, we have $|V(F_1'')| \neq |V(T_3'')|$ or $|V(F_3'')| \neq |V(T_1'')|$. However, this implies $W_H(uw) \ncong W_K(uw)$, a contradiction.

Now assume $|V(F_1)| \neq |V(T_3)|$. Then by the choice of F, $|V(F_3)| \neq |V(T_1)|$. Since $W_G(uw) \cong W_H(uw)$, $|V(F')| = |V(F)|$, and either $|V(F_1')| = |V(F_1)|$ and $|V(T_3')| = |V(T_3)|$, or $|V(F_1')| = |V(T_3)|$ and $|V(T_3')| = |V(F_1)|$. Suppose $|V(F_1')| = |V(F_1)|$ and $|V(T_3')| = |V(T_3)|$. Then $W_G(T) \cong W_H(T')$, and it follows from Lemma (2.5) that $W_G(F) \cong W_H(F')$. Note that each isomorphism $\pi : W_G(F) \to W_H(F')$ must send u, w, u_1 to u, w, u_1, respectively. This implies $H \cong G$. Hence, we may assume $|V(F_1')| = |V(T_3)|$ and $|V(T_3')| = |V(F_1)|$. Then, any isomorphism $\pi : W_G(uw) \to W_H(uw)$ must send u, w to w, u, respectively, because the sum of the sizes of facial cycles of G containing u is different from that of facial cycles of H containing u. Since $|V(F_3')| = |V(F_3)| \neq |V(T_1)| = |V(T_1')|$, π must send F_1, F_3, T_1, T_3 to T_1', T_3', F_1', F_3'. However, this implies $|V(F_1)| = |V(T_1')| = |V(T_1)| = |V(F_1')| = |V(T_3)|$, a contradiction. □

The next result shows the reconstructibility of 5-connected planar graphs with isolated 5-vertices. Since its proof is similar to (but much more complicated than) the proof of the above result, we only give a sketch of its proof; a more detailed proof is available from the authors upon request.

(4.2) Proposition. *Let G be a 5-connected plane graph, and assume that G has an isolated 5-vertex. Then G is reconstructible.*

Proof. Let H be a reconstruction of G. We shall prove $H \cong G$. Let v be an isolated 5-vertex, let $u_1, \ldots, u_5$ denote the neighbors of v in counter-clockwise order around v, and let F_i denote the facial cycle of G containing $\{v, u_i, u_{i+1}\}$, for all $1 \leq i \leq 5$. In this proof, for any integer j we let $\alpha_j = \alpha_k$, where $1 \leq k \leq 5$ is the smallest integer equal to j modulo 5. (For example, $u_6 = u_1$ and $u_{-1} = u_4$.) For each $1 \leq i \leq 5$, let $F_i, F_{i-1}, F_{i,1}, F_{i,2}, \ldots, F_{i,j_i}$ be the facial cycles of G containing u_i in counter-clockwise order around u_i. For convenience, let $f_i := |V(F_i)|$ and $f_{i,j} := |V(F_{i,j})|$ for $1 \le i \le 5$ and $1 \le j \le j_i$.

Using Lemmas (2.3), (2.4), and (2.5), we can show that we may assume:

(1) $f_1 = f_2 = f_3 = f_4 = f_5 = 3$;

(2) $f_{1,1}, f_{2,1}, f_{3,1}, f_{4,1}$ and $f_{5,1}$ are all distinct.

Note that the two facial cycles in $W_G(vu_5)$ sharing edges with F_4 and F_5 but not containing v have sizes $f_{5,1}$ and $f_{1,1}$. On the other hand, H may be obtained from $G - u_1$ by adding u_1 into the new face of $G - u_1$. We use the same notation as in G for the vertices of H, and let F'_i denote the facial cycles of H containing $\{v, u_j, u_{j+1}\}$, $1 \le j \le 5$. Because of (1) and (2) and by Lemma (2.4), we can show that if $H \not\cong G$ then there is no edge wheel $W_H(vu_i)$ with $i \in \{1, \ldots, 5\}$ such that the two facial cycles in $W_H(vu_i)$ sharing an edge with F'_{i-1} and F'_i but not containing v have sizes $f_{5,1}$ and $f_{1,1}$. So if $H \not\cong G$ then $W_G(vu_5) \not\cong W_H(vu_i)$ for any $1 \le i \le 5$, contradicting Lemma (2.4). □

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