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On Tight Components and Anti-Tight Components

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Abstract A graph $G = (V, E)$ is called factor-critical if $G \neq \emptyset$ and $G - v$ has a perfect matching for every vertex $v \in V(G)$. A factor-critical graph G is tight (anti-tight, respectively) if for any $v \in V(G)$, any perfect matching M in $G - v$, and any $e \in M$, $|N(v) \cap V(e)| \neq 1$ ($|N(v) \cap V(e)| \neq 2$, respectively), where $N(v)$ denotes the neighborhood of v and $V(e)$ denotes the set of vertices incident with e. A graph G is minimally anti-tight if G is anti-tight but $G - e$ is not anti-tight for every $e \in E(G)$. In this paper, we prove that a connected graph is tight iff every block of the graph is an odd clique, and that every minimally anti-tight graph is triangle-free.

Keywords Perfect matching · Tight component · Anti-tight component

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1 Introduction

Let G be a graph and H a component of G. For any $v \in V(G)$, we use $N_G(v)$ to denote the set of neighbors of v in G. For any $e \in E(G)$, we use $V_G(e)$ to denote the set of vertices of G incident with e. (If understood, we omit the reference to G.) If $G \neq \emptyset$ and $G - v$ has a perfect matching for every $v \in V(G)$, then G is called *factor-critical*. Factor-critical graphs have been extensively studied in the past [2,4,5,7,8]. We say that H is *odd* if $|H|$ is odd and that H is a *tight component* (respectively, *anti-tight component*) if

- H is factor-critical, and
- $|N(v) \cap V(e)| \neq 1$ (respectively, $|N(v) \cap V(e)| \neq 2$) for any $v \in V(H)$, any perfect matching M in $H - v$, and any $e \in M$.

When $H = G$, we simply say that G is tight (respectively, anti-tight).

Lee et al. [6] used tight components to solve a problem of Bollobás and Scott [1,9] about the dependence on minimum degree of bounds on judicious bisections. A *block* in a graph G is a maximal connected subgraph that contains no cut vertex. Hence, if a block of G is not 2-connected, then it must be induced by a cut edge of G. A complete subgraph of a graph is usually called a *clique*. It is easy to see that odd cliques are tight. Lee et al. [3] observed that if every block of a connected graph G is an odd clique then G is tight, and mentioned that it is not clear if every tight component is of this form. In this note, we answer this question in the affirmative.

Theorem 1 *A connected graph is tight iff every block of the graph is an odd clique.*

It is apparent that if a graph G is factor-critical and triangle-free, then G is anti-tight. But it is not true that every anti-tight graph G is triangle-free (see Fig. 1). It seems difficult to characterize the anti-tight graphs. A connected graph G is *minimally anti-tight* if G is anti-tight but $G - e$ is not anti-tight for every $e \in E(G)$. Concerning the minimally anti-tight graphs, we have

Theorem 2 *Every minimally anti-tight graph G is triangle-free.*

Let G be a graph and M a matching in G. A vertex v is called *matched* if it is incident with an edge in M, and otherwise v is called *unmatched*. An *M-alternating path* is a path in which the edges belong alternatively to $E(G) \setminus M$ and M. An *M-augmenting path* is an alternating path that starts from and ends at unmatched vertices. Let H be a subgraph of G, we use $M \Delta H$ to denote the symmetric difference of M and $E(H)$. In general, we follow the terminology in [3].

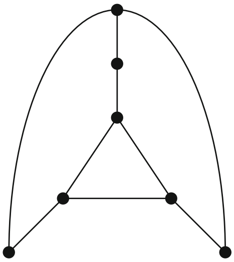

Fig. 1 An anti-tight graph with a *triangle*

2 Proof of Theorem 1

Lemma 1 *Let G_1 and G_2 be subgraphs of a connected graph G such that $G = G_1 \cup G_2$ and $|V(G_1) \cap V(G_2)| = 1$. Then G is tight iff both G_1 and G_2 are tight.*

Proof Let $x \in V(G_1) \cap V(G_2)$. Suppose G is tight. Let $v \in V(G_1)$. Then $G - v$ has a perfect matching, say M. Note that $|V(G_1)|$ is odd and thus the edge in M incident with x belongs to G_1 when $x \neq v$; so $M \cap E(G_1)$ is a perfect matching in $G_1 - v$. Now let M_1 be a perfect matching in $G_1 - v$. Since $G - x$ has a perfect matching, $G_2 - x$ has a perfect matching, say M_2. Then $M := M_1 \cup M_2$ is a perfect matching in $G - v$. Thus, since G is tight, for any $e \in M_1$, $|N_G(v) \cap V(e)| \neq 1$; so $|N_{G_1}(v) \cap V(e)| = |N_G(v) \cap V(e)| \neq 1$. Hence, G_1 is tight. Similarly, we can show that G_2 is tight.

Now assume both G_1 and G_2 are tight. Let $v \in V(G)$. Then $v \in V(G_i)$ for some $i \in 1, 2$ and, without loss of generality, let $v \in V(G_1)$. Let M_1 be a perfect matching in $G_1 - v$, and M_2 a perfect matching in $G_2 - x$; so $M_1 \cup M_2$ is a perfect matching in $G - v$. Now for any perfect matching M in $G - v$, $M \cap E(G_1)$ is a perfect matching in $G_1 - v$ while $M \cap E(G_2)$ is a perfect matching in $G_2 - x$. Moreover, for any $e \in M \cap E(G_1)$, $|N_G(v) \cap V(e)| = |N_{G_1}(v) \cap V(e)| \neq 1$ (as G_1 is tight). Now suppose $e \in M \cap E(G_2)$. If $v \neq x$ then $|N_G(v) \cap V(e)| = 0 \neq 1$. So assume $v = x$. Then since G_2 is tight, $|N_G(v) \cap V(e)| = |N_{G_2}(v) \cap V(e)| \neq 1$. Hence, G is also tight. □

Lemma 2 *Let G be a 2-connected tight graph. Then G is a complete graph with odd order.*

Proof Suppose to the contrary that G is not a complete graph. Then we can find three vertices x, y, z such that $yx \in E(G)$, $yz \in E(G)$ but $xz \notin E(G)$. As $G - y$ has a perfect matching, say M, we let $xx_1 \in M$ and $zz_1 \in M$. Since $|\{x, x_1\} \cap N_G(y)| \neq 1$ and $|\{z, z_1\} \cap N_G(y)| \neq 1$, $x_1 y \in E(G)$ and $z_1 y \in E(G)$. Note that $G' := G - \{x, y, z\}$ has no perfect matching, otherwise, $G - \{x\}$ has a perfect matching containing yz and $|N_G(x) \cap \{y, z\}| = 1$, a contradiction. Then $x_1 z_1 \notin E(G)$. If $xz_1 \in E(G)$, $G - z$ has a perfect matching $M' = (M - \{xx_1, zz_1\}) \cup \{xz_1, yx_1\}$ and $|N_G(z) \cap \{x, z_1\}| = 1$, a contradiction. Hence, $xz_1 \notin E(G)$. Similarly, $zx_1 \notin E(G)$.

Suppose $G - y$ has an M-alternating path $P' = xy_2 \cdots y_k z$. If $xy_2 \in M$ and $y_k z \in M$, then $y_2 = x_1$ and $y_k = z_1$, hence, $M \Delta yxy_2 \cdots y_k$ is a perfect matching in $G - z$ and $|N_G(z) \cap \{y, x\}| = 1$, a contradiction. If $xy_2 \in M$ and $y_k z \notin M$, then $y_2 = x_1$ and $y_k \neq z_1$, clearly, $M \Delta y_2 \cdots y_k z z_1 y$ is a perfect matching in $G - x$ and $|N_G(x) \cap \{z_1, y\}| = 1$, a contradiction. If $xy_2 \notin M$ and $y_k z \in M$, then we get a similar contradiction. Thus, $xy_2 \notin M$ and $y_k z \notin M$. Then $y_2 \neq x_1$ and $y_k \neq z_1$. So $M \Delta xy_2 \cdots y_k z z_1 y$ is a perfect matching in $G - x_1$ and $|N_G(x_1) \cap \{z_1, y\}| = 1$, a contradiction.

Thus, $G - y$ does not contain any M-alternating path from x to z. If x_1 and z are linked by a path P_1 in $G - \{x, y\}$, let $P = xx_1 + P_1 = xx_1 v_3 \cdots v_i v_{i+1} v_{i+2} \cdots z$ and assume that $xx_1 v_3 \cdots v_i v_{i+1}$ is an M-alternating path, $v_{i-1} v_i \in M$ but $v_i v_{i+1}, v_{i+1} v_{i+2} \notin M$ for some i. Choose P so that $xx_1 v_3 \cdots v_i v_{i+1}$ is longest and

$v_{i+1}v_{i+2}\cdots z$ is shortest. Then the matched vertex v'_{i+1} with v_{i+1} is not in P as $xx_1, v_3v_4, \cdots, v_{i-1}v_i \in M$ and $v_{i+1}v_{i+2}\cdots z$ is shortest. If $v_iv'_{i+1} \notin E(G)$, then $M\Delta yxx_1v_3\cdots v_i$ is a perfect matching in $G - v_i$ and $|N_G(v_i) \cap \{v_{i+1}, v'_{i+1}\}| = 1$, a contradiction. Thus $v_iv'_{i+1} \in E(G)$. Now $xx_1v_3\cdots v_iv'_{i+1}\ v_{i+1}v_{i+2}\cdots z$ is a path contradicts the choice of P, as it has a longer M-alternating section starting from x.

Hence x_1 and z are not linked by any path in $G - \{x, y\}$. Then since G is 2-connected, there is a path $P = xv_2\cdots z$ in $G - y$ such that $x_1 \notin P$. Let $P = xv_2\cdots v_iv_{i+1}v_{i+2}\cdots z$ such that $xv_2v_3\cdots v_iv_{i+1}$ is an M-alternating path, $v_{i-1}v_i \in M$ but $v_iv_{i+1}, v_{i+1}v_{i+2} \notin M$. Choose P so that $xv_2\cdots v_iv_{i+1}$ is longest and $v_{i+1}v_{i+2}\cdots z$ is shortest. Then the matched vertex v'_{i+1} with v_{i+1} is not in P as $v_{i+1} \neq x_1$ and $v_2v_3, \cdots, v_{i-1}v_i \in M$ and $v_{i+1}v_{i+2}\cdots z$ is shortest. If $v_iv'_{i+1} \notin E(G)$, then $M\Delta yx_1xv_2\cdots v_i$ is a perfect matching in $G - v_i$ and $|N_G(v_i) \cap \{v_{i+1}, v'_{i+1}\}| = 1$, a contradiction. Thus $v_iv'_{i+1} \in E(G)$. So $xv_2v_3\cdots v_iv'_{i+1}v_{i+1}v_{i+2}\cdots z$ is a path contradicts the choice of P, as it has a longer M-alternating section starting from x. □

A simple corollary of Lemma 1 is the observation of Lee, Loh and Sudakov that if every block of a graph G is an odd clique then G is a tight component. Clearly, Lemma 1 and Lemma 2 imply Theorem 1 which says the converse.

3 Proof of Theorem 2

Assume to the contrary that G is minimally anti-tight and contains a triangle $xyzx$. If $G - xz - v$ has a perfect matching for every $v \in V(G)$, every such matching M is also a perfect matching in $G - v$. Since G is anti-tight, $|N_{G-xz}(v) \cap V(e)| \leq |N_G(v) \cap V(e)| \leq 1$ for every $e \in M$. Hence, $G - xz$ is anti-tight, a contradiction.

Thus, there is a vertex $u \in V(G)$ such that $G - xz - u$ contains no perfect matching. $G - y$ has a perfect matching M and $xz \notin M$. This implies $u \neq y$. Let $\widetilde{M}$ be a perfect matching in $G - u$. Since $G - xz - u$ contains no perfect matching, $xz \in \widetilde{M}$. Let G' be the multigraph with $V(G') = V(G)$ and whose edges are edges in M and $\widetilde{M}$. It is easy to see that $d_{G'}(u) = d_{G'}(y) = 1$ and for any $v \in V(G) - \{u, y\}$, $d_{G'}(v) = 2$. So there is a path $P = uv_1v_2\cdots v_ty$ in G' between u and y, and the length of P is even. If $xz \notin P$, then xz is an edge of an even cycle C which is a component of G', and $\widetilde{M}\Delta C$ is a perfect matching of $G - xz - u$, a contradiction. Hence, $xz \in P$, and let $\{v_i, v_{i+1}\} = \{x, z\}$. Since $v_iv_{i+1} \notin M$, the length of $uv_1\cdots v_i$ is odd and $C_1 := v_i\cdots v_ty + yv_i$ is an even cycle in G. Now $\widetilde{M}\Delta C_1$ is a perfect matching of $G - xz - u$, a contradiction. □

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